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ANALYTICAL MODEL SOLUTION OF THE GROUNDWATER FLOW EQUATION

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Abstract
This paper presents an analytical model solution for the prediction of the one-dimensional (1D) time-dependent groundwater flow profile in an unconfined system. Groundwater level can be estimated by using the proposed solution with several input data, such as permeability layer thicknesses, specific yield. This hydraulic charge prediction problem is modeled as a boundary value problem governed by the classic heat diffusion equations. The solution technique employs the separation of variables method and the result are compared to the 2 implicit numerical solutions of CrankNicholson and FTCS, the solution displays a reasonable groundwater flow head in contexts of sand-gravel aquifer during different time periods.
Key words: Variable Separation, Groundwater Equation, Diffusion Equation, Porous Media

1 Introduction
Today's water resources are increasingly threatened due to natural, household, industrial and agricultural pollution and intensive consumption. Hydrogeology is therefore a subject of prime importance for a society like ours: it must help to properly manage drinking water resources and ensure the lowest possible pollution levels for its inhabitants. It is for these reasons that it is necessary to provide the means to correctly predict the behavior of water flows and the transport of contaminants in the subsoil.

As for several problems and phenomena of physics which can be modeled by partial differential equations (PDE). The phenomena of groundwater flows are modeled by PDEs, associated with boundary conditions and initial conditions.

The groundwater equation that is governed through Darcy law and the continuity equation were the subject of a set of research. Among the first researches that concerned with this equation [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. Initially, these researches focused on trying to understand the behavior of groundwater and its flow mechanisms in porous media. These researches have focused to find solutions to the groundwater equation, the researchers analyzed the mechanisms of evolution and regular groundwater flow regeneration in the aquifers. as a result, it has been proposed and developed a set of analytical solutions of the groundwater flow equation as a model of forecasting and simulating the dynamic behavior of groundwater based on boundary conditions [11, 12, 13, 14, 4].

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In the same context, some recherche has focused on finding in obtaining analytical solutions to the linear Boussinesq equation by adopting the uniform recharge of the rainfall rate, therefore these solutions are exploited to estimate the groundwater levels change and drainage flow [13, 14, 30]. Other research focused on researching analytical solutions for the same equation, taking into account the hypothesis of temporal variation of the level of rainfall [25, 31, 22, 23, 24, 16].

Some other research has worked on the Laplace Transform method with the aim of developing an analytical solution to express the distribution of groundwater level [3, 26, 27, 28, 29].

The present study focused on method of Separation of Variables for solving the groundwater equation using Darcy's law as a theoretical basis and applied the principle of mass conservation (continuity equation) to govern the groundwater flow.

The remainder of this paper is organized as follows: Section 2 introduces the governing of groundwater equation. Section 3 and 4 proposes the analytical method of one-dimensional diffusion operator in rectangular coordinates using method of separation of variable. Section 5 demonstrates the feasibility of the analytical solution using synthetic examples and the compared with a two numerical method, it comes in a FTCS method (Forward Time Centered Space [30]) and the CrankNicholson method [31] . Section 6 provides discussions the result. Finally, Section 7 summarizes and concludes this work.

2 Derivation of Groundwater Equation

We begin by deducting of partial differential equations that occur in describing flows in porous media phenomena. The general groundwater flow equation is deducted from Darcy's law and from the continuity equation. The law of conservation of mass for transient flow shows that the net rate of change of density is exactly opposite to net rate of change of volume itself (V), in other words, the net rate of penetration of a fluid in a control volume is exactly equal to the net rate of change of storage of the mass of fluid in the same control volume.

$$Inflow\ NetRate = Inflow - Outflow = Storage\ Change\ Rate \tag{2.1}$$

This is equivalent:

$$Inflow\ NetRate = -div(\rho\vec{v}) = -\left(\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right) \tag{2.2}$$

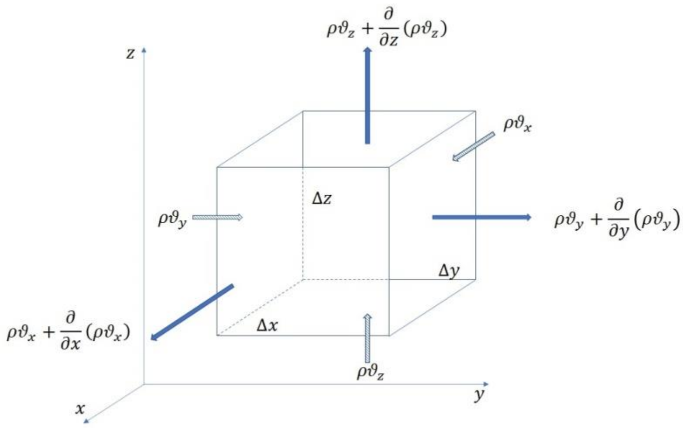

Figure 1: Groundwater control volume

It should be noted that In steady-state flow, the change in storage within the control volume is equal to zero. In transient flow, the change in storage It should violate zero, Thus, the previous equation 2 will become:

$$-div(\rho\vec{v}) = -\frac{\partial(\rho v_x)}{\partial x} - \frac{\partial(\rho v_y)}{\partial y} - \frac{\partial(\rho v_z)}{\partial z} = \frac{\partial(\rho\eta)}{\partial t} \tag{2.3}$$

where η is the porous media porosity. The $\frac{\partial(\rho\eta)}{\partial t}$ term is the time rate of fluid mass change per unit volume of the control volume (the term has dimensions M/L^3T). We consider that it is a saturated porous medium. When using the chain-rule, the $\frac{\partial(\rho\eta)}{\partial t}$ term will become:

$$\frac{\partial(\rho\eta)}{\partial h}\frac{\partial h}{\partial t} = \frac{\partial(\rho\eta)}{\partial t} \tag{2.4}$$

When it is a transient saturated flow, the change rate in fluid storage in the control volume is related to the change rate in the hydraulic head. so the $\frac{\partial(\rho\eta)}{\partial t}$ term becomes:

$$\eta\frac{\partial\rho}{\partial h} + \rho\frac{\partial\eta}{\partial h} = \frac{\partial(\rho\eta)}{\partial t} \tag{2.5}$$

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The $\rho\frac{\partial\eta}{\partial h}$ term in equation 5 is the water mass produced by the compression or expansion of the porous media, the $\eta\frac{\partial\rho}{\partial h}$ term is the water mass produce by the compression or expansion of the fluid. for the saturated case, if the porosity increases $\frac{\partial\eta}{\partial h} > 0$, or if the fluid density increase $(\frac{\partial\rho}{\partial h} > 0)$. the water can enter the control volume.

we pose now α as the porous media compressibility and β as the fluid compressibility, and σ_e as a change in effective stress (Compression or expansion of the porous media).

For the saturated case:

$$d\sigma_e = -\rho g d\Phi \tag{2.6}$$

Were Φ is pressure head. since $d\Phi = (h - z) = dh - dz$ then:

$$d\sigma_e = -\rho g dh \tag{2.7}$$

Now we can define the compressibility of porous media α

$$\alpha = -\frac{dV_f}{V}\frac{1}{d\sigma_e} = \frac{d\eta}{d\sigma_e} \tag{2.8}$$

Where V is the fluid volume and it is the control volume. let combining equations 7 and 8 we write :

$$\frac{d\eta}{dh} = \alpha\rho g \tag{2.9}$$

We can define the fluid compressibility β as:

$$\beta = \frac{dV_f}{V_f}\frac{1}{dp} \tag{2.10}$$

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We note p the pressure of the fluid. the expression of the change in pressure is given by :

$$dp = \rho g d\Phi = \rho g dh \qquad (2.11)$$

And with $dV_f/V_f = d\rho/\rho$, we can write equation 10 as:

$$\beta = \frac{d\rho}{\rho}\frac{1}{\rho g dh} \qquad (2.12)$$

or

$$\frac{d\rho}{dh} = \rho^2 g\beta \qquad (2.13)$$

After substituting of equations 9 and equation 13 into equation 4, we've got:

$$\frac{\partial}{\partial t}(\rho n) = (\rho\frac{\partial n}{\partial h} + n\frac{\partial \rho}{\partial h})\frac{\partial h}{\partial t} = (\rho^2 g\alpha + n\rho^2 g\beta)\frac{\partial h}{\partial t} \qquad (2.14)$$

Now we can define the specific storage S_s as:

$$S_s = \rho g(\alpha + n\beta) \qquad (2.15)$$

The dimensions of the specific storage S_s are L^{-1}, this term is representing the water volume that an aquifer unit volume releases from storage for a unit decline in hydraulic head.

After substituting of equation 15 into equation 14, we've got:

$$\frac{\partial}{\partial t}(\rho n) = \rho S_s\frac{\partial h}{\partial t} \qquad (2.16)$$

After substituting of equation 16 into equation 3, we've got:

$$-\frac{\partial}{\partial x}(\rho v_x) - \frac{\partial}{\partial y}(\rho v_y) - \frac{\partial}{\partial z}(\rho v_z) = \rho S_s\frac{\partial h}{\partial t}$$

(2.17)

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We suppose that the density ρ it's a constant, equation 17 becomes:

$$\rho(-\frac{\partial}{\partial x}v_x - \frac{\partial}{\partial y}v_y - \frac{\partial}{\partial z}v_z) = \rho S_s \frac{\partial h}{\partial t} \quad (2.18)$$

We simplify equation 18 by eliminating ρ from both sides of the equation, we have compensated in v according to Darcy law ($v_x = K_x \frac{\partial h}{\partial x}$, $v_y = K_y \frac{\partial h}{\partial y}$,$v_z = K_z \frac{\partial h}{\partial z}$).

$$\frac{\partial}{\partial x}(K_x \frac{\partial h}{\partial x}) + \frac{\partial}{\partial y}(K_y \frac{\partial h}{\partial y}) + \frac{\partial}{\partial z}(K_z \frac{\partial h}{\partial z}) = S_s \frac{\partial h}{\partial t} \quad (2.19)$$

The equation 19 represents the transient saturated-flow equation, when K_x,K_y,K_z are homogeneous, they will be constants and equation 19 can be written: :

$$K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} + K_z \frac{\partial^2 h}{\partial z^2} = S_s \frac{\partial h}{\partial t} \quad (2.20)$$

If the porous media is also isotropic$K_x = K_y = K_z = K$, equation 20 is written:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{S_s}{K} \frac{\partial h}{\partial t} \quad (2.21)$$

In the case of a confined aquifer, equation 21 is written:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T} \frac{\partial h}{\partial t} \quad (2.22)$$

Where b is constant thickness, $S = S_s b$ and $T = Kb$.

3 One-Dimensional groundwater equation

We begin by examining the last groundwater flow phenomena (Diffusion), that are treated similarly with a linear diffusion partial differential equation.

$$\frac{\partial}{\partial x}(K_x\frac{\partial}{\partial x}h(x,y,z,t))+\frac{\partial}{\partial y}(K_y\frac{\partial}{\partial y}h(x,y,z,t))+\frac{\partial}{\partial z}(K_z\frac{\partial}{\partial z}h(x,y,z,t)) \tag{3.1}$$

$$=S_s\frac{\partial}{\partial t}h(x,y,z,t)-q(x,y,z,t)$$

We examine the following partial differential equations in one dimension that have linear operators.

$$(\frac{dK(x)}{dx})(\frac{\partial}{\partial x}h(x,t))+K(x)(\frac{\partial^2}{\partial x^2}h(x,t))=S_s(\frac{\partial h(x,t)}{\partial t})-q(x,t) \tag{3.2}$$

$h(x,t)$ denotes the spatial-time dependent groundwater head, $K(x)$ denotes the hydraulic conductivity, S_S denotes the specific storage, and $q(x,t)$ denotes the time rate of the input/output source into the medium volume. If $q(x,t)=0$, then there are no flow sources in the system and this non homogeneous partial differential equation reduces to its corresponding homogeneous equation :

$$(\frac{dK(x)}{dx})(\frac{\partial}{\partial x}h(x,t))+K(x)(\frac{\partial^2}{\partial x^2}h(x,t))=S_s(\frac{\partial h(x,t)}{\partial t}) \tag{3.3}$$

We consider that the medium is uniform. which means that the coefficient of permeability K is spatially invariant, we can write to them as simple constants. Then, equation 3 is simplified by:

$$k(\frac{\partial^2}{\partial x^2}h(x,t))=\frac{\partial h(x,t)}{\partial t} \tag{3.4}$$

Where k, the groundwater flow diffusivity or hydraulic diffusivity of the medium, are L^2/T:

$$k=\frac{K}{S_s} \tag{3.5}$$

This partial differential equation for groundwater flow phenomena is characterized that the hydraulic gradient (slope) is found from the first derivative and the curve concavity the is given from the second derivative with respect to the spatial variable x. In addition, the time rate of change, or excavation rate $(\frac{\partial h(x,t)}{\partial t})$, is given as the first derivative with respect to the time variable t.

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4 Groundwater flow equation. Separation method

In this level, we will proceed to solve the groundwater flow equation (3.4) by the variable separation method:

$$k(\frac{\partial^2}{\partial x^2}h(x,t)) = \frac{\partial h(x,t)}{\partial t} \tag{4.1}$$

We note that x represents the position in the one-dimensional medium (an aquifer) that we can identify with the interval $[0, L]$. The hydraulic height in this aquifer at time t and at location x is $h(x,t)$. A typical problem is to consider that the distribution of the hydraulic height over the entire length of the aquifer is known at time $t = 0$ (initial condition) and that the flow of groundwater through the ends $x = 0$ and $x = L$ are given values (boundary conditions). Therefore we can imagine that the hydraulic height is determined for $x \in (0, L)$ and $t > 0$. The conditions imposed on the ends are often of the form:

$$h(0,t) = 0 \text{ or, } \frac{\partial h(0,t)}{x} = 0 \text{ or } \frac{\partial h(0,t)}{x} = 0 = ah(0,t)$$

$$h(L,t) = 0 \text{ or, } \frac{\partial h(L,t)}{x} = 0 \text{ or } \frac{\partial h(L,t)}{x} = 0 = -ah(L,t)$$

Where $a > 0$ is also a physical constant. Let we find a solution h on following typical problem:

1 *Find h such that:*

$$\frac{\partial h(x,t)}{\partial t} = k(\frac{\partial^2}{\partial x^2}h(x,t)) pour\ 0 < x < L\ and\ t > 0, \tag{4.2}$$

$$h(0,t) = 0\ et\ \frac{\partial}{\partial x}h(L,t) = 0, for\ t > 0 \tag{4.3}$$

$$h(x,0) = \varphi(x), for\ 0 < x < L \tag{4.4}$$

where $\varphi: [0, L] \to R$ is the initial distribution of hydraulic height, assumed to be known. It would be reasonable to assume that φ also meets the conditions imposed at the ends of the aquifer. They are compatibility conditions:

$$\varphi(0) = \frac{\partial \varphi}{\partial x}(L) = 0 \tag{4.5}$$

The variable separation method for the problem can be presented in the following two steps:

• step 1: we are looking for solutions of problem (2) and (3) (Separation of variables) which are of the form:

$$h(x, t) = f(x)g(t), but\ h \not\equiv 0 \tag{4.6}$$

• step 2: Overlay, ie; we seek to find a sum of solutions of the form 6 that satisfies condition 4. Note that such a sum still checks (2) and (3).

(1) A function of the form (6) is a solution of the equation (2) if $f \in C^2((0, L)), g \in C^1((0, \infty))$ and

$$f(x)g'(t) = kf''(x)g(t)\ pour\ 0 < x < L\ and\ t > 0. \tag{4.7}$$

At the level where $h(x, t) = f(x)g(t) \neq 0$, this is written as follows:

$$\frac{g\prime(t)}{g(t)} = k\frac{f\prime\prime(x)}{f(x)} \tag{4.8}$$

And therefore there is a constant $\lambda \in \mathbb{R}$ such that:

$$\frac{g\prime(t)}{g(t)} = k\frac{f\prime\prime(x)}{f(x)} = \lambda \tag{4.9}$$

For all points $(x, t) \in (0, L)(0, \infty)$ such as $h(x, t) \neq 0$. There is at least one point $y \in (0, L)$ such as $f(y) \neq 0$. Sine $g \not\equiv 0$, there is an interval $(t_1, t_2) \subset (0, \infty)$ such as $g(t) \neq 0$ for all $t \in (t_1, t_2)$ and therefore $u(y, t) \neq 0$ for these values of t. In particular,

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$$g'(t) = \lambda(t)\ for\ all\ t \in (t_1, t_2) \quad (4.10)$$

Which implies that there is a constant $A \neq 0$ such as $g(t) = Ae^{\lambda t}$for all $t \in (t_1, t_2)$.The continuity of g on $(0, \infty)$ means that $g(t_1) = Ae^{\lambda t_1} \neq 0$ and that $g(t_2) = Ae^{\lambda t2} \neq 0$ .We can easily deduce that $g(t) \neq 0$ on all $(0. \infty)$ and that:

$$g(t) = Ae^{\lambda t_1}\ pour\ tout\ t > 0. \quad (4.11)$$

On the other hand, condition (3) becomes:

$$f(0)g(t) = f'(L)g(t) = 0\ for\ all\ t > 0. \quad (4.12)$$

So we have to impose that:

$$f(0) = f'(L) = 0$$

we then see to satisfy condition (3) if $h \not\equiv 0$. the function f must be a non-trivial solution, i.e. $f \not\equiv 0$, of the boundary problem:

$$f''(x) = \frac{\lambda}{k} f(x)\ for\ \leq x \leq L, and\ f(0) = f'(L) = 0 \quad (4.13)$$

We will see that this is only possible for certain very particular values of the constant, called eigenvalues of the boundary problem (13). Let us calculate all the solutions of (13). The general solution to the differential equation is

$$f(x) = \begin{cases} Pe^{-\sqrt{\frac{\lambda}{k}}x} + Qe^{\sqrt{\frac{\lambda}{k}}x} & si\ \lambda > 0 \\ P + Qx & si\ \lambda = 0 \\ P\ cos\sqrt{\frac{|\lambda|}{k}}x + Q\ sin\sqrt{\frac{|\lambda|}{k}}x & si\ \lambda < 0 \end{cases} \quad (4.14)$$

Where P and Q are arbitrary real constants which must be chosen so that f satisfies the boundary conditions.

Case $\lambda > 0$. The condition $f(0) = 0$ is satisfied if and only if $P + Q = 0$, which means that f must be of the form $f(x) = 2Qsinh\left(\sqrt{\frac{\lambda}{k}}\mathrm{x}\right)$ and so $f'(x) = \sqrt{\frac{\lambda}{k}}2Qcosh\left(\sqrt{\frac{\lambda}{k}}x\right)$.To ensure that f satisfies the condition $f'(L) = 0$, choose Q such that $\sqrt{\frac{\lambda}{k}}2Qcosh\left(\sqrt{\frac{\lambda}{k}}L\right) = 0$ and the only solution is to ask $Q = 0$ because $cosh(y) > 0$ for all $y \in R$. But if $P = Q = 0$, $f \equiv 0$ and therefore problem 13 admits no non-trivial solution when $\lambda > 0$.

Case $\lambda = 0$. In that case, $f(0) = P$ and $f'(L) = Q$. So there is no non-trivial solution of (13) in this case either.

Case $\lambda < 0$. The condition $f(0) = 0$ equals to $P = 0$ and therefore f must be of the form $f(x) = Qsin\sqrt{\frac{|\lambda|}{k}}x$. The condition $f'(L) = 0$ becomes $\sqrt{\frac{|\lambda|}{k}}Qcos\left(\sqrt{\frac{|\lambda|}{k}}L\right) = 0$ and we can choose $Q \neq 0$ As long as $cos\left(\sqrt{\frac{|\lambda|}{k}}L\right) = 0$. That is, problem 13 admits non-trivial solutions if and only if the constant λ is such that $cos\left(\sqrt{\frac{|\lambda|}{k}}L\right)$. hence:

$$\sqrt{\frac{|\lambda|}{k}}L \in \left\{\frac{2n+1}{n}\pi : n \in \mathbb{Z}\right\} and\ \lambda < 0 \tag{4.15}$$

We pose:

$$\lambda_n = -k(n + \tfrac{1}{2})^2(\tfrac{\pi}{L})^2\ for\ n \in \mathbb{N} \tag{4.16}$$

we see that we obtain all the eigenvalues of problem (13). The set of all eigenvalues:

$$\sigma = \{\lambda_n : n \in \mathbb{N}\} \tag{4.17}$$

give the spectrum of the problem (12) and the function:

$$f_n(x) = Qsin\sqrt{\frac{|\lambda_n|}{k}}x = Qsin\frac{(n+\frac{1}{2})\pi}{L}x, where\ Q \neq 0 \tag{4.18}$$

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Is an eigenfunction associated with the eigenvalue λ_n.

Then for summary of the 1st step, Equations (2) and (3) admit non-trivial solutions of the form (6) if and only if f is a eigenfunction of the boundary problem (13) and g is of the form (11) where is the eigenvalue associated with f. So the solutions of problem $(2)$and (3) of form (6) are:

$$h_n(x,t) = Q_n sin(\sqrt{\frac{|\lambda_n|}{k}}x)e^{\lambda_n t} \tag{4.19}$$

where $\lambda_n \in \sigma$ and Q_n is an arbitrary real constant, not zero.

(2) Superposition: The form of equations (2) (3) allows the superposition of solutions. We can easily see that:

$$h(x,t) = \sum_{n=0}^{m} Q_n sin\,(\sqrt{\frac{|\lambda_n|}{k}}x)e^{\lambda_n t} \tag{4.20}$$

Is also a solution of equations (2) and (3) whatever $m \in \mathbb{N}$ and the constants Q_n. We now try to choose m and N so that u satisfies condition (4). Which means:

$$\varphi(x) = \sum_{n=0}^{m} Q_n sin\sqrt{\frac{|\lambda_n|}{k}}x, pour\ tout\ x \in (0,L) \tag{4.21}$$

This is possible if and only if

$$\varphi \in ev\{f_n : n \in \mathbb{N}\} = \{\sum_{n=0}^{m} Q_n\, f_n : m \in \mathbb{N}\ and\ Q_n \in \mathbb{R}\} \tag{4.22}$$

Where f_n is an eigenfunction given by (18). When $\varphi =\in ev\{f_n : n \in \mathbb{N}\}$, the coefficients Q_n are determined by the following calculation. First, note that

$$\int_0^L sin\,(\sqrt{\frac{|\lambda_n|}{\mathrm{k}}}x)sin(\sqrt{\frac{|\lambda_j|}{k}}x)dx = \begin{pmatrix} 0 & si\,n \neq j \\ \frac{L}{2} & si\,n \neq j \end{pmatrix} \tag{4.23}$$

Admitting that $\varphi \in ev\{f_n : n \in \mathbb{N}\}$ and therefore that it can be expressed by (21), we multiply (21) by $sin(\sqrt{\frac{|\lambda_j|}{c}}$ and then we integrate from 0 to L. We find that:

$$\int_0^L \varphi(x)\, sin\left(\sqrt{\frac{|\lambda_j|}{k}}x\right) dx = \int_0^L \sum_{n=0}^{m} sin\,(\sqrt{\frac{|\lambda_n|}{k}}x) sin(\sqrt{\frac{|\lambda_j|}{k}}x) dx \tag{4.24}$$

$$= \sum_{n=0}^{m} Q \int_0^L sin\,(\sqrt{\frac{|\lambda_n|}{k}}x) sin(\sqrt{\frac{|\lambda_j|}{k}}x) dx \tag{4.25}$$

$$= Q_j \frac{L}{2}\; si\; 0 \le j \le m \tag{4.26}$$

So

$$Q_j = \frac{2}{L} \int_0^L \varphi(x)\, sin(\sqrt{\frac{|\lambda_j|}{k}}x) dx, if\; 0 \le j \le m \tag{4.27}$$

while $Q_k = 0$ for all $k > m$. i.e., if $\varphi \in ev\{f_n : n \in \mathbb{N}\}$ so:

$$\varphi(x) = \sum_{n=0}^{\infty} Q_n\, sin(\sqrt{\frac{|\lambda_n|}{k}}x) dx, for\; all\; x \in (0, L), where \tag{4.28}$$

$$Q_n = \frac{2}{L} \int_0^L \varphi(x)\, sin(\sqrt{\frac{|\lambda_n|}{k}}x) dx, for\; all\; n \in \mathbb{N} \tag{4.29}$$

Because there is $m \in N$ such as $\varphi(x) sin(\sqrt{\frac{|\lambda_j|}{c}}x) dx = 0$ for all $n > m$.

The method of separating variables solves problem (2), (3) and (4) for functions $\varphi \in ev\{f_n : n \in \mathbb{N}\}$ where f_n is a eigenfunction of the boundary problem (13). the solution obtained is

$$h(x,t)=\sum_{n=0}^{\infty} sin\left(\left[\left(\frac{|\lambda_n|}{k}\right)^{1/2} x\right]\right) Q_n e^{\lambda_n t} \quad (4.30)$$

Since $\lambda_n=-k(n+\frac{1}{2})^2\left(\frac{\pi}{L}\right)^2$, the equation can be written in this formula:

$$h(x,t)=\sum_{n=0}^{\infty} sin\,(\alpha\pi x)Q_n e^{-k[\alpha\pi]^2 t}$$

Where $\alpha=\frac{(2n+1)}{2L}$, and the $Q_n$is:

$$Q_n=\frac{2}{L}\int_0^L sin\left(\left[(\beta)^{1/2}x\right]\right)\varphi(x)dx, \forall n\in\mathbb{N} \quad (4.31)$$

Where $\beta=\frac{|\lambda_n|}{k}$, $Q_n$also can be written :

$$Q_n=\frac{2}{L}\int_0^L sin\,(\alpha\pi x)\varphi(x)dx \quad (4.32)$$

The sum in (30) is over for everything $\varphi\in ev\{f_n: n\in\mathbb{N}\}$.

Remark 2. *Note that all functions $\varphi\in ev\{f_n: n\in\mathbb{N}\}$ are infinitely differentiable on $[0,L]$ and check the compatibility conditions (5). Solution (30) is infinitely differentiable over $[0,L][0,inf)$ because it is the sum of a finite number of functions of this kind. Given a function φ which is infinitely differentiable on $[0,L]$, we can determine whether or not it belongs $\varphi\in ev\{f_n: n\in\mathbb{N}\}$ by calculating the integrals (31). So $\varphi\in ev\{f_n: n\in\mathbb{N}\}$ if and only if all except a finite number of these integrals are zero. These same calculations give the coefficients in solution (30).*

5 Simulation of solution

Consider the case of a $1D$ flow problem on an unconfined aquifer that a river and a lake run parallel to each other (figure 2) with $L=500m$ apart. They fully penetrate aquifer with a hydraulic conductivity $K=10\ m/day$, and specific yield $S_s=30\%$. Let us run the groundwater head solution in different boundary conditions cases, and as we change the hydraulic diffusivity (k) in time and space (example 1, example 2, example 3, example 4). In order to demonstrates the feasibility of the analytical solution given by method of separation of variable, we are compared it by a two numerical profile simulated using the CrankNicholson implicite (5.1) method and the Forward Time Centered Space method $(eq5.2)$.

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This method consists in replacing the second derivative $\frac{\partial^2 h}{\partial x^2}$ in the equation (4.1) by the average of its discrete representations at times n and $n+1$.

$$\left[\frac{\partial^2 h}{\partial x^2}\right]_i^{n+1} = \frac{1}{h^2}\left[\frac{1}{2}\left(h_{i+1}^{n+1} - 2h_i^{n+1} + h_{i-1}^{n+1}\right) + \frac{1}{2}\left(h_{i+1}^{n} - 2h_i^{n} + h_{i-1}^{n}\right)\right] \tag{5.1}$$

And for the Forward Time Centered Space (FTCS) or forward/backward space method is an implicit single stage finite difference method that can used for numerically solving the heat equation and similar parabolic partial differential equations. This scheme is unconditionally stable. then the equation (4.1) can be represented by the flowing scheme:

$$h_i^{n+1} = h_i^n + \alpha\left(h_{i+1}^n - 2h_i^n + h_{i-1}^n\right) \tag{5.2}$$

With $\alpha = \frac{k\Delta t}{\Delta x^2}$

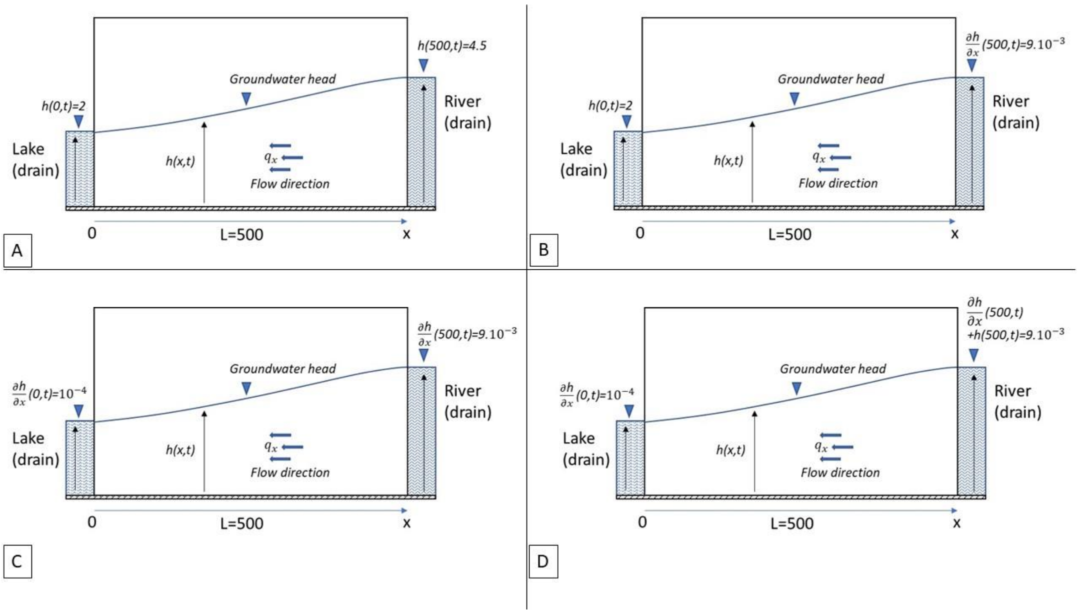

Figure 2: Groundwater flow conceptual model on unconfined, horizontal aquifer with different types of boundary conditions

Exemple *1:*

This example corresponds to the following mathematical problem with nonhomogeneous Dirichlet boundary conditions (Figure .2A)

$$\begin{cases} k(\frac{\partial^2}{\partial x^2}h(x,t)) = \frac{\partial h(x,t)}{\partial t} & ,0 \le x \le L, 0 \le t \le T \\ h(0,t) = 2, h(L,t) = 4.5 & ,0 \le t \le T \\ h(x,0) = (2 + \frac{x^3}{5\times 10^7}) & ,0 \le x \le L \end{cases} \quad (5.3)$$

Following the solution (4.30) derived in section 4, the solution of problem (5.3) is show in figure 3, the figure provides a good match solution at different time $t = 10, t = 20, t = 30, t = 40$.

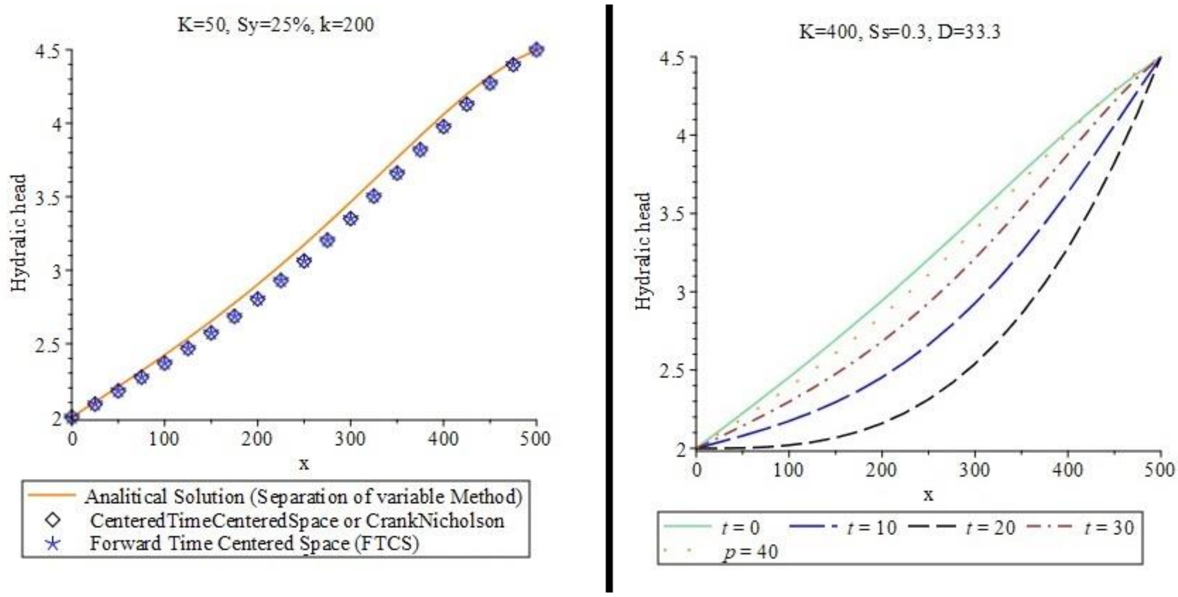

Figure 3: (a): Comparison of the evaluated exact solution correspond of example 1 and figure 2A with implicit numerical methods of CrankNicholson and FTCS; (b) evaluated exact solution in various time.

Exemple *2:*

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This example corresponds to the following mathematical problem with a Dirichlet boundary conditions type in the lake side, and a Neumann boundary type conditions in the river limit side (Figure .2B) :

$$\begin{cases} k(\frac{\partial^2}{\partial x^2}h(x,t)) = \frac{\partial h(x,t)}{\partial t} & ,0 \leq x \leq L, 0 \leq t \leq T \\ h(0,t) = 2, \frac{\partial}{\partial x}h(L,t) = 9.10^{-3} & ,0 \leq t \leq T \\ h(x,0) = (2 + \frac{x^3}{5\times 10^7}) & ,0 \leq x \leq L \end{cases} \quad (5.4)$$

Following the solution (4.30) derived in section 4, the solution of problem (5.4) is show in figure 4, the figure provides a good match solution at different time $t = 10, t = 20, t = 30, t = 40$.

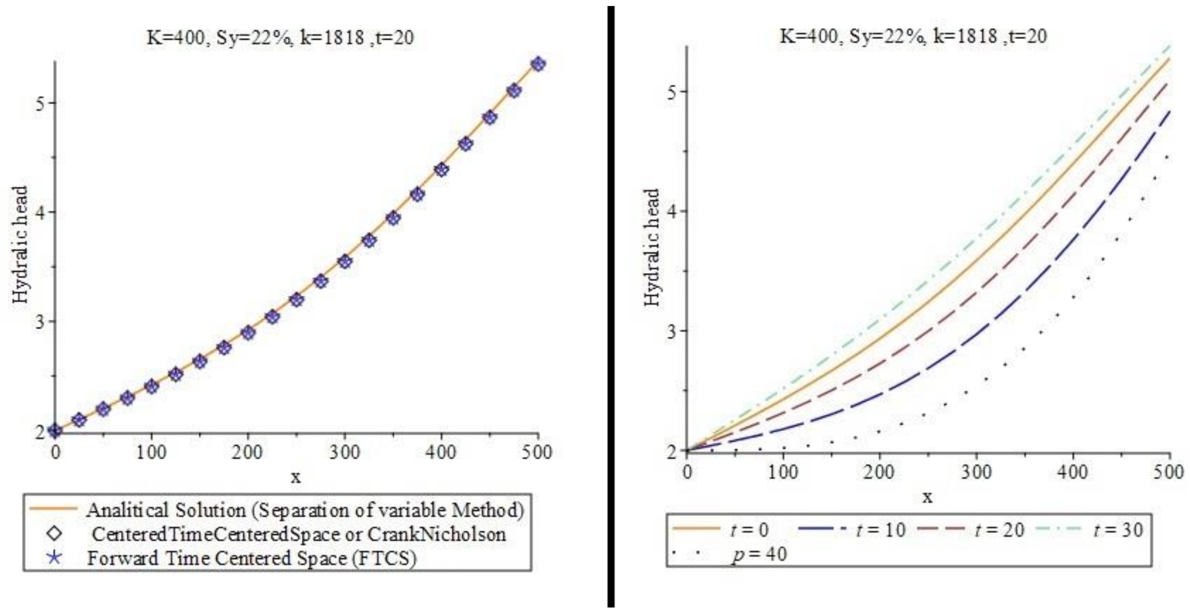

Figure 4: (a): Comparison of the evaluated exact solution correspond of example 2 and figure 2B with implicit numerical methods of CrankNicholson and FTCS; (b) evaluated exact solution in various time.

Exemple *3:*

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This example corresponds to the following mathematical problem with nonhomogeneous Neumann boundary conditions (Figure 2C) :

$$\begin{cases} k(\frac{\partial^2}{\partial x^2}h(x,t)) = \frac{\partial h(x,t)}{\partial t} & ,0 \leq x \leq L, 0 \leq t \leq T \\ \frac{\partial}{\partial x}h(0,t) = 0.0001, \frac{\partial}{\partial x}h(L,t) = 0.009 & ,0 \leq t \leq T \\ h(x,0) = (2 + \frac{x^3}{5\times 10^7}) & ,0 \leq t \leq L \end{cases} \tag{5.5}$$

Following the solution (4.30) derived in section 4, the solution of problem (5.5) is show in figure 5, the figure provides a good match solution at different time $t = 10, t = 20, t = 30, t = 40$.

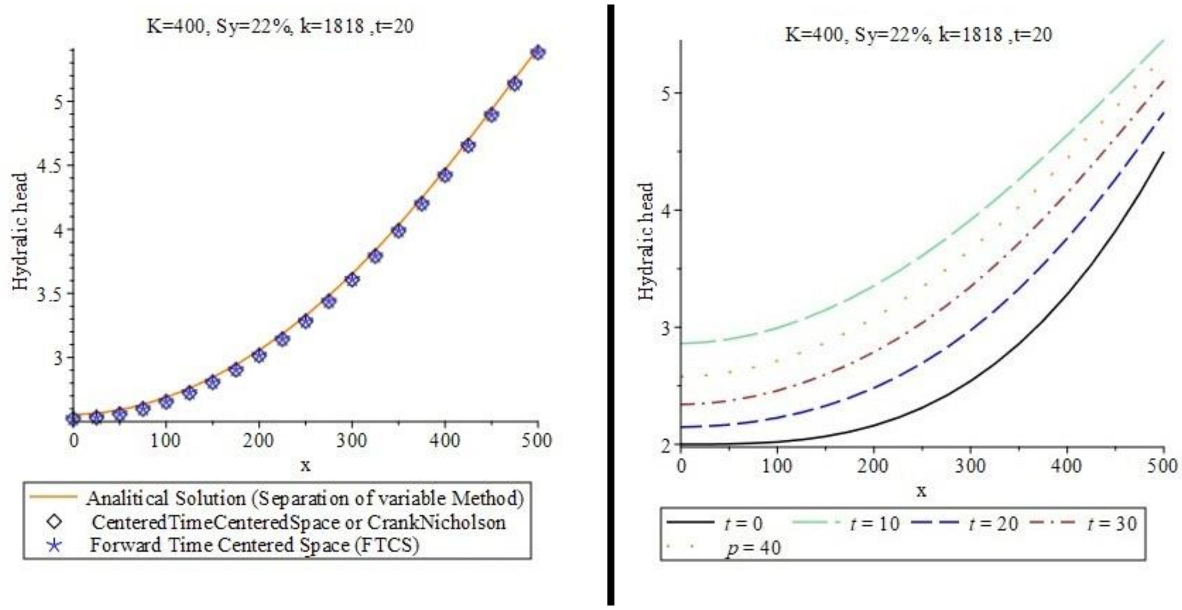

Figure 5: (a): Comparison of the evaluated exact solution correspond of example 3 and figure 2C with implicit numerical methods of CrankNicholson and FTCS; (b) evaluated exact solution in various time.

Exemple *4:*

This example corresponds to the following mathematical problem with a Neumann boundary conditions type in the lake limit side and a Mixed boundary conditions type in the river limit side (Figure 2D) :

$$\begin{cases} k(\frac{\partial^2}{\partial x^2}h(x,t)) = \frac{\partial h(x,t)}{\partial t} & ,0 \le x \le L, 0 \le t \le T \\ \frac{\partial}{\partial x}h(0,t) = 10^{-4}, h(L,t) + \frac{\partial}{\partial x}h(L,t) = 4.5 & ,0 \le t \le T \\ h(x,0) = h(x,0) = (2 + \frac{x^3}{5\times10^7}) & ,0 \le x \le L \end{cases} \tag{5.6}$$

Following the solution (4.30) derived in section 4, the solution of problem (5.6) is show in figure 6, the figure provides a good match solution at different time $t = 10, t = 20, t = 30, t = 40$.

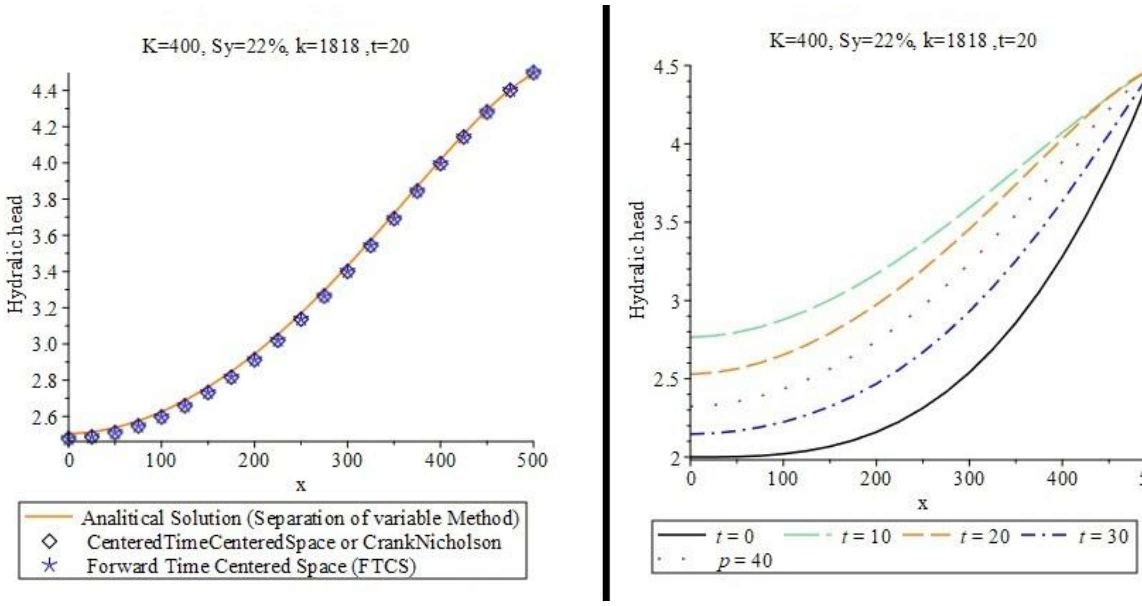

Figure 6: (a): Comparison of the evaluated exact solution correspond of example 4 and figure 2D with implicit numerical methods of CrankNicholson and FTCS; (b) evaluated exact solution in various time.

6 Discussion

Analytical solution for the prediction of the one-dimensional $(1D)$ time-dependent groundwater flow profile in an unconfined system evaluated for a setting corresponding case to $L = 500m, K = 10, K = 50, K = 100$, $K = 400m/d$, $k = 31818m^2/d$, $S_y = 22\%$, the solution use a uniform domain in different time step length.

The $Figure.3$ to $Figure.6$ depicts results obtained at 4 different limit conditions according to equation (4.30). in the 4 cases of simulation, the results showing note that the solution that was produced by the method of separation of the variable is an acceptable solution, as well as that in all the four cases in which the solution was applied, it was found that there is a match between the solution produced by the separation of variable method and with the other two numerical methods of CrankNicholson and FTCS.

The module can simulate the same solutions that were given by the CrankNicholson and FTCS methods. It can then be concluded that the solution is given by the separation of the variable method when applied inhomogeneous medium, taking into account the normal boundary conditions, the solution presented can reproduce the behavior of the groundwater in a very acceptable way.

7 Conclusion

In this work, we introduce an analytical solution for a homogenous porous media. Using the method of separation of variables, this analytical solution accurately reproduces the results from the finite difference numerical method given by CrankNicholson and FTCS. We test the proposed solution in four examples, considering not only the constant head (Dirichlet conditions) boundary condition but also a specific flux (condition de Neumann) and mixed condition (Cauchy condition) . We compared the analytical method and therefore the two numerical methods of CrankNicholson and FTCS. using a sand-gravel medium characteristics of permeability, hydraulic dispersivity and specific yield. The analytical solution and therefore the CrankNicholson and FTCS numerical method are in good agreement within the four example cases. The proposed method is valid for the overall homogenous horizontal unconfined aquifer, including not only fluid flow in porous media but also another physical problems.

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