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A PROPOSED METHOD FOR SOLVING A SPECIAL CASE OF MULTI-OBJECTIVE FRACTIONAL PROGRAMMING PROBLEM

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Abstract

This paper presents a special case of multi-objective linear fractional programming (MOLFP) problem as a new assumption to transform MOLFP to the multi-objective linear programming (MOLP) problem. The new assumption is, the denominators of the fractional objective functions in the MOLFP problem are the same. In the beginning, we introduced the linear fractional programming (LFP) problem with the solution method to illustrate the proposed method. Our proposed is based on Charnes and Cooper transformation, which is transforming the fractional objective functions to MOLP problem. The Weighted sum method is considered in this paper to solve the MOLP problem after Charnes and Cooper transformation. To illustrate the proposed method, a numerical example is presented.

Keywords: linear fractional programming, multi-objective linear fractional programming, multi-objective linear programming, Charnes and Cooper transformation, weighted sum method.

1. INTRODUCTION

In several applications of nonlinear programming a function is to be maximized or minimized which involves one or several ratios of functions. Such optimization problems are commonly called fractional programs, abbreviating the term 'fractional functionals program' initially suggested by Charnes and Cooper in their classic paper in (1962). Charnes and Cooper replaced any “linear fractional programming problem ’’, at most, two straightforward linear programming problems that differ from each other by only a change in sign in the functional and in one constraint. Also the variable transformations to be utilized will be simpler than the square-root transformations. Their transformations are also homomorphisms; from which follows the globality of local optima with linear fractional functionals.

Linear Fractional Programming (LFP) problem is a mathematical programming problem where the objective function is the ratio of two linear functions subject to the constraints with linear equalities or inequalities. The Hungarian mathematicians, (Martos and Whinston, 1964) developed linear fractional programming problem in the 1960s. LFP problem is applied when the constraints and objective functions are deterministic in nature. After Charnes and Cooper showed that, a linear programming problem with a linear fractional objective function could be solved by solving most two ordinary linear programming problems. In addition, they showed that where it is known a priori that the denominator of the objective function has a unique sign in the feasible region, only one problem need be solved. In the present note it is shown that if

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a finite solution to the problem exists, only one linear programming problem must be solved. This is because the denominator cannot have two different signs in the feasible region, except in ways which are not of practical importance. Stanley Zionts in (1968) purposed the note off to show that for programming problems with linear fractional functionals," only one linear programming problem (and not two) must be solved if a finite solution to the problem exists.

In 1985**,** Nykowski I. and Zołkiewski Z. Formulated the multiple objective fractional programming problem and introduced A concept of multiple objective programing problem corresponding to the multiple objective fractional programming problem with some relations between those problems are examined. Based on these results, they proposed a compromise procedure for the multiple objective linear fractional programming problem.

Do sang kim in (2005) considered multi-objective fractional programming problem with generalized invexity. He formulated an equivalent multi-objective fractional programming problem by using a modification of the objective function due to Antczak (2003). Also, he gave relations between a multi-objective fractional programming problem and an equivalent multi-objective fractional problem which has a modified objective function. After 5 years in (2010), Lotfi F. H. et al. Applied a geometrical interpretation and a linear programming approach was achieved to test weak efficiency. Also, they constructed a linear programming approach in order to test strong efficiency for a given weakly efficient point.

Odior A. O. and Oyawale F. A. in (2012) presented a new approach for solving a fractional linear programming problem in which the objective function is a linear fractional function, while the constraint functions are in the form of linear inequalities. Their approach adopted was based mainly upon solving the problem algebraically using the concept of duality and partial fractions.

A new solution to the Multi-objective Linear Fractional Programming Problem is Proposed by Güzel N. in (2013). His proposed solution was based on a theorem that deals with non-linear fractional programming with a single objective function and studied in the work by Dinkelbach (1967) and reduced the linear fractional programming to a linear programming problem. In the same year, Sulaiman N. A. and Abdulrahim B. K. (2013) Proposed a new transformation technique for solving multi-objective linear fractional programming problem to single-Objective linear fractional programming problem, through a new method using mean and median, and then solve the problem by modifying the simplex method. Jain S. (2014) Derived Gauss elimination technique of inequalities for numerical solution of multi-objective fractional programming problem by using the concept of bounds. His method was quite useful because the calculations involved a simple and takes less time.

In 2020**,** Kumari P. Considered the problem of finding the optimal value of the ratio of two linear functions subject to some linear restrictions. He assumed that the feasible region of the solution was a convex polyhedron and the optimal value exists at an extreme point of the convex polyhedron. The basic idea of his proposed method was the search of an optimal solution by moving from one extreme point to the other of the feasible region until the optimal solution was reached. All the extreme points were first obtained by plotting the graphs of the

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corresponding linear inequalities and then, only those were considered which are feasible. At each feasible extreme point, thus obtained, the value of the objective function was calculated, and the one, at which the objective function has the optimal value, was the desired solution.

A complex multi-objective fractional programming problem (CMFP) is Considered by Huang T. Y. and Ho S. C. in (2021). They established the necessary optimality conditions of problem CMFP in the sense of Pareto optimality and derived its sufficient optimality conditions using generalized convexity. Also, they constructed the parametric dual problem to the primal problem CMFP and their duality theorems.

Our motivation in this paper is to propose a method for solving a special case of multi-objective linear fractional programming problem. We assume that the denominator of the fractional objective functions is the same.

The rest of the paper is prepared as follows. In Section 2, introduce the linear fractional programming problem with how to solve it. In Section 3, an MOLFP problem is discussed. In Section 4, introduce a special case of MOLFP as a new assumption. In Section 5, proposed a solution method for an MOLFP problem. In section 6, numerical example for illustrating the solution of the proposed method. Finally, concluding remarks are given in Section 7.

2. LINEAR FRACTIONAL PROGRAMMING

Mathematical Programming (MP) has been widely used to investigate different aspects of any systems in recent decades. Among the different MP techniques, Fractional Programming (FP), which is similar to any model has fractional in its objective function according to its mathematical background, is a well-known technique for optimizing the efficiency of decision-making objective (Amini F. A. et al., 2010). In FP the goal is to optimize the ratio between physical and/or economic functions (Gómez T. et al., 2006), which are linear combinations of decision variables. In a general form, the mathematical structure of a single objective linear fractional program with n decision variables and m constraints can be written as (Goedhart M. H. and Spronk J., 1995):

$$\text{Max}\, z = \frac{c^T x + \alpha}{d^T x + \beta}$$

s.t

$$x \in S = \{x \in R^n | Ax \le b; x \ge 0; b \in R^m\} \qquad (1)$$

The numerator and denominator of the objective fraction are real functions defined on R^n, with the decision variables vector x, technical coefficients vectors c, d and scalar constants α, β. The right hand side vector, b, of the constraints is defined on R^m, so technical coefficients, A, form an $m * n$ matrix.

To find the optimum solution for this problem, a new variable (y) need to be introduced under an additional assumption in which the denominator of the above quotient is strictly positive throughout the feasible set of solutions (see the details in Charnes and Cooper paper).

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$y = xt$ and $t = \frac{1}{d^T x + \beta}$ (2)

Using this transformation, the original fractional problem is changed to an ordinary linear programming problem with an additional constraint as follows:

$$\text{Max } z = c^T y + \alpha$$
$$\text{s.t}$$
$$Ay - bt \leq 0 \qquad (3)$$
$$d^T y + \beta t = 1$$
$$y, t \geq 0$$

Based on this transformation, if (y', t') is an optimum solution of the problem, then $\left(x' = \frac{y'}{t'}\right)$ Will be an optimum solution of the original fractional problem (Charnes A. and Cooper W., 1962; Goedhart M. H. and Spronk J. 1995)).

3. MULTI-OBJECTIVE LINEAR FRACTIONAL PROGRAMMING PROBLEM

An MOLFP problem is defined as follows (Solomon M. et al., 2021)

$$\text{Maximize } \{g(x) = (g_1(x), g_2(x), \dots, g_k(x))\}$$
$$\text{s.t } Ax \leq b \qquad (4)$$
$$x \geq 0$$

Where:

$S = \{x \in R^n \,|Ax \leq b, x \geq 0, b \in R^m\}$, is the Feasible Set in Decision Space.

A is an $m \times n$ matrix, $x \in R^n$ and $b \in R^m$; $(b \geq 0), k \geq 2$.

$$g_i(x) = \frac{c_i^T x + \alpha_i}{d_i^T x + \beta_i} = \frac{N_i(x)}{D_i(x)} \;;\; c_i^T, d_i^T \in \mathbb{R}^n \;;\; \alpha_i, \beta_i \in \mathbb{R}; \text{ for all } i = 1,2,\dots,k$$

and $D_i(x) = d_i^T x + \beta_i > 0$, for all $i = 1,2,\dots,k$, for all $x \in S$.

A solution $\bar{x} \in S$ is an efficient solution of the problem MOLFP if and only if there is no $x \in S$ such that $g_i(x) \geq g_i(\bar{x})$ for all $i = 1,2,\dots,k$ and $g_i(x) > g_i(\bar{x})$ for at least one i.

Note that, for vectors and y; $x \geq y$ *implies* $x_i \geq y_i$ *for each* i , $x \geq y$ *implies* $x_i \geq y_i$ *for* i *and* $x_r > y_r$ *for at least one* $i = r$ *and* $x > y$ *implies* $x_i > y_i$ *for each i* .

4. A SPECIAL CASE OF MOLFP PROBLEM

A special case of MOLFP problem is considered in this paper. The denominators of the fractional objective functions in MOLFP are the same, solving this problem will be tarnsformating the MOLFP problem to the multi-objective linear programming. Then, any approaches which solving MOLP will be good to solve MOLP after transformation.

The special case of MOLFP problem is defined as follows:

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$$Max\ F_1 = \frac{c_i^T x + \alpha_i}{d^T x + \beta}$$
$$Max\ F_2 = \frac{c_i^T x + \alpha_i}{d^T x + \beta}$$
$$\vdots$$
$$Max\ F_k = \frac{c_i^T x + \alpha_i}{d^T x + \beta}$$

s.t

$$Ax \leq b \qquad (5)$$
$$x \geq 0$$

Where:

$S = \{x \in R^n \mid Ax \leq b, x \geq 0, b \in R^m\}$, is the Feasible Set in Decision Space.

$A\ is\ an\ m \times n\ matrix,\ x \in R^n\ and\ b \in R^m;\ (b \geq 0), k \geq 2.$

$c_i^T, d^T \in \mathbb{R}^n;\ \alpha_i, \beta \in \mathbb{R};\ \forall\ i = 1,2,\ldots,k\ and\ d^T x + \beta > 0,\ \ \forall\ x \in S.$

5. A PROPOSED METHOD FOR TRANSFORMING MOLFP TO THE MOLP PROBLEM

First of all, we must make sure the denominators of the objective functions are the same. After that to solve this problem do these steps: Consider an MOLFP problem illustrated in the model (5)

Step 1: a new variable (y) (in equations (2)) needs to be introduced under an additional assumption in which the denominator of the quotient is strictly positive throughout the feasible set of solutions.

Step 2: after using this transformation, the original MOLFP problem is changed to an ordinary MOLP problem with an additional constraint as follows:

$$Max\ F_1 = c_i^T y + \alpha_i$$
$$Max\ F_2 = c_i^T y + \alpha_i$$
$$\vdots$$
$$Max\ F_k = c_i^T y + \alpha_i$$

s.t

$$Ay - bt \leq 0 \qquad (6)$$
$$d^T y + \beta t = 1$$
$$y, t \geq 0$$

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Based on this transformation, if (y', t') Is an optimum solution of the problem, then $\left(x' = \frac{y'}{t'}\right)$ Will be an optimum solution of the original fractional problem.

Step 3: solve the problem (6) as an MOLP problem with any approaches solves this problem successfully.

Step 4: Find an optimal solution of the linear programming problem after transforming MOLP obtained in step 3. This solution is an efficient solution of the MOLFP problem.

In the next section, numerical example will be introduced to illustrate our idea of the proposed method.

6. NUMERICAL EXAMPLE

Consider the following multi-objective linear fractional programming problem with the same denominator: (this example was modified with the new assumption; Güzel, N. 2013)

$$Max\ F_1 = \frac{-3x_1 + 2x_2}{x_1 + x_2 + 3}$$

$$Max\ F_2 = \frac{7x_1 + x_2}{x_1 + x_2 + 3}$$

s.t

$$x_1 - x_2 \geq 1$$

$$2x_1 + 3x_2 \leq 15 \qquad (7)$$

$$x_1 \geq 3$$

$$x_1, x_2 \geq 0$$

Step 1: let $t = \frac{1}{x_1+x_2+3}$ and $y_1 = tx_1 \ , y_2 = tx_2$

Step 2: the model after transformation will be:

$$Max\ F_1 = -3y_1 + 2y_2$$

$$Max\ F_2 = 7y_1 + y_2$$

s.t

$$y_1 - y_2 - t \geq 0$$

$$2y_1 + 3y_2 - 15t \leq 0 \qquad (8)$$

$$y_1 - 3t \geq 0$$

$$y_1 + y_2 - 3t = 1$$

$$y_1, y_2 \geq 0$$

Step 3: weighted sum method is considered for solving problem (8)

$$0.5\ (-3y_1 + 2y_2) + 0.5\ (7y_1 + y_2) = \ -1.5y_1 + y_2 + 3.5y_1 + 0.5y_2$$

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Then, the new objective function will be:

$$Max\ W = 2y_1 + 1.5y_2$$

s.t

$$y_1 - y_2 - t \geq 0$$

$$2y_1 + 3y_2 - 15t \leq 0 \qquad (9)$$

$$y_1 - 3t \geq 0$$

$$y_1 + y_2 - 3t = 1$$

$$y_1, y_2 \geq 0$$

Step 4: solve the problem (9) as a linear programming problem with any programs like Tora or WinQSB.

The optimal solution will be:

$$y_1 = \frac{5}{7} \quad , \quad y_2 = 0 \quad , \quad t = \frac{2}{21}$$

Then, the optimal solution of the original fractional problem is:

$$\mathbf{x_1 = 7.5 \quad , \quad x_2 = 0 \quad , F_1 = -\frac{15}{7} = -2.14 \quad , F_2 = 5}$$

If we solved problem (7) by the proposed method of Güzel, N. [9] the solution will be:

$$\mathbf{x_1 = 3 \quad , \quad x_2 = 0 \quad , \quad F_1 = -1.5 \quad , \quad F_2 = 3.5}$$

In table 1: It is observed that the optimal objective values $F_1(x), F_2(x)$ Obtained due to the proposed method are considerably closer and comparable to that of due to Guzel N. Proposed method.

Table 1: comparative results

Objective Function	**Max $F_1(x)$**	**Max $F_2(x)$**
Our proposed	-2.14	5
Guzel N. proposed	-1.5	3.5

7. CONCLUSION

In this paper, a special case of multi-objective linear fractional programming is considered. In the practical field, it's possible that the fractional objective functions have the same denominator in economics, engineering and organization problems etc.., A simple and efficient method by Charnes and Cooper is used to transform each fractional objective function to linear objective function. After that, we have the MOLP problem obtained after the transformation, which is solved by the weighted sum method. These methods as a methodology for the special case problem has solved it successfully after transforming the MOLFP into an MOLP problem. The solution achieved due to the proposed methods is compared with the Nuran Guzel proposed method which verifies the effectiveness of its performance and considerably closer

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and comparable to him. A numerical example is given to illustrate our motivation for considering a special case of MOLFP problem.

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