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SOME APPLICATIONS OF QUADRUPLE HYPERGEOMETRIC FUNCTIONS IN FUNCTIONS SPACES

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Abstract:

In this paper we have obtained a set of formulae of generating functions Lauricella's F_A, Horn's H_4, Saran's F_E and F_G, Pandey's G_B and Exton's K_{10} in terms of Exton's quadruple hypergeometric functions $K_5, K_9, K_{10}, K_{13}, K_{16}$ and associated with quadruple function D_5.

Keywords: Hypergeometric functions, Horn's functions, Saran's functions, Exton's functions, etc.

MSC: 33C05, 33C20, 33C65.

INTRODUCTION:

In 1971, 1972, 1973, 1976, 1984, Exton was the first famous mathematician who studied on certain hypergeometric differential system, certain hypergeometric functions of four variables, some integral representations and transformations of hypergeometric functions of four variables, on certain hypergeometric differen system II, multiple hypergeometric functions and applications and hypergeometric partial differential system of third order respectively (see [1-7]).

In 1992 and 1993, Mathai defined Jacobians of matrix transformations I and hypergeometric functions of several matrix arguments (see [8-9]). Many works dealing with transformation formulae for Exton's quadruple hypergeometric functions and their associated properties has been done (see [10]-[24]). Similarly, in 1984, Srivastava and Manocha (see [25]) defined the following generating function

$$\sum_{g=0}^{\infty}\binom{f+g}{g}\frac{(\mu)_g}{(1+\in+f)_g}\xi_{f+g}^{(\in)}(a)\,x^g=\binom{\in+f}{f}(1-x)^{-\mu}\,\varphi_2\left[-f,\mu;1+\in;a,\frac{ax}{x-1}\right] \qquad (1)$$

Where f is positive integer, $\xi_{f+g}^{(\in)}(a)$ and φ_2 denotes generalized Laguerre polynomials and Humbert's functions.

In 1920, Humbert (see [26]) defined seven functions and some of which are the limiting form of Appell's functions and they are as follows:-

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$$\varphi_1[\alpha;\beta;\gamma;a,\mathrm{b}]=\sum_{f,g=0}^{\infty}\frac{(\alpha)_{f+g}(\beta)_f}{(\gamma)_{f+g}}\frac{a^f.b^g}{f!.g!} \quad (2)$$

$$\varphi_2[\alpha;\beta;\gamma;a,b]=\sum_{f,g=0}^{\infty}\frac{(\alpha)_f(\beta)_g}{(\gamma)_{f+g}}\frac{a^f\,b^g}{f!.g!} \quad (3)$$

$$\varphi_3[\alpha;\beta;a,b]=\sum_{f,g=0}^{\infty}\frac{(\alpha)_f}{(\beta)_{f+g}}.\frac{a^f}{f!}.\frac{b^g}{g!} \quad (4)$$

$$\zeta_1[\alpha;\beta;\gamma,r;a,b]=\sum_{f,g=0}^{\infty}\frac{(\alpha)_{f+g}(\beta)_f}{(\gamma)_f(r)_g}.\frac{a^f}{f!}.\frac{b^g}{g!} \quad (5)$$

$$\zeta_2[\alpha;\beta,\gamma;a,b]=\sum_{f,g=0}^{\infty}\frac{(\alpha)_{f+g}}{(\beta)_f(\gamma)_g}.\frac{a^f}{f!}.\frac{b^g}{g!} \quad (6)$$

$$\chi_1[\alpha,\beta,\gamma;r,a,b]=\sum_{f,g=0}^{\infty}\frac{(\alpha)_f(\beta)_g(\gamma)_f}{(r)_{f+g}}\frac{a^f}{f!}\frac{b^g}{g!} \quad (7)$$

The Saran functions (see [25]) are defined as follows:

$$F_4:F_E[\alpha,\alpha,\alpha,\beta,\gamma,\gamma;r,s,w;a,b,c]=\sum_{f,g,\ell=0}^{\infty}\frac{(\alpha)_{f+g+\ell}(\beta)_f(\gamma)_{g+\ell}}{(r)_f(s)_g(w)_\ell}\frac{a^f}{f!}.\frac{b^g}{g!}.\frac{c^\ell}{\ell!},$$

Where $j+\left(k^{\frac{1}{2}}+x^{\frac{1}{2}}\right)=1,$ (8)

$$F_8:F_G[\alpha,\alpha,\alpha;\beta,\gamma,r;s,w,w;\ a,b,c]=\sum_{f,g,\ell=0}^{\infty}\frac{(\alpha)_{f+g+\ell}(\beta)_f(\gamma)_g(r)_\ell}{(s)_f(w)_{g+\ell}}\frac{a^f}{f!}.\frac{b^g}{g!}.\frac{c^\ell}{\ell!},$$

Where $j+k=1=j+x$ (9)

$$F_3:F_K[\alpha,\beta,\beta,\gamma,r,\gamma;s,w,u;a,b,c]=\sum_{f,g,\ell=0}^{\infty}\frac{(\alpha)_f(\beta)_{g+\ell}(\gamma)_{f+\ell}(r)_g}{(s)_f(w)_g(u)_\ell}.\frac{a^f}{f!}.\frac{b^g}{g!}.\frac{c^\ell}{\ell!},$$

$$(1-j)(1-k)=x \quad (10)$$

$$F_{11}:F_M[\alpha,\beta,\beta,\gamma,r,\gamma;s,w,w;a,b,c]=\sum_{f,g,\ell=0}^{\infty}\frac{(\alpha)_f(\beta)_{g+\ell}(\gamma)_{f+\ell}(r)_g}{(s)_g(w)_{g+\ell}}.\frac{a^f}{f!}.\frac{b^g}{g!}.\frac{c^\ell}{\ell!},$$

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$$j + x = 1 = k \tag{11}$$

$$F_6 : F_N[\alpha,\beta,\gamma,r,s,r;w,u,u;a,b,c] = \sum_{f,g,\ell=0}^{\infty} \frac{(\alpha)_f(\beta)_g(\gamma)_\ell(r)_{f+g}(s)_g}{(w)_f(u)_{g+\ell}} \cdot \frac{a^f}{f!} \cdot \frac{b^g}{g!} \cdot \frac{c^\ell}{\ell!},$$

$$(1-j)k + (1-k)x = 0 \tag{12}$$

In 1963, Pandey (see [10]) transformed the integral representations of F_F and F_G into two interesting hypergeometric series which is similar to Horn's type (see [27]-[29])

$$G_A[\alpha,\beta,\gamma;r;a,b,c] = \sum_{f,g,\ell=0}^{\infty} \frac{(\alpha)_{g+\ell-f}(\beta)_{f+\ell}(\gamma)_g}{(r)_{g+\ell-f}} \cdot \frac{a^f}{f!} \cdot \frac{b^g}{g!} \cdot \frac{c^\ell}{\ell!}$$

$$g + \ell - f \tag{13}$$

it provides a generalization of Appell's function F_1 and Horn's function G_1 and G_2; and

$$G_B[\alpha,\beta_1,\beta_2,\beta_3;r;a,b,c] = \sum_{f,g,\ell=0}^{\infty} \frac{(\alpha)_{g+\ell-f}(\beta_1)_f(\beta_2)_g(\beta_3)_\ell}{(r)_{g+\ell-f}} \cdot \frac{a^f}{f!} \cdot \frac{b^g}{g!} \cdot \frac{c^\ell}{\ell!} \tag{14}$$

it provides the generalization of Appell's function F_1 and Horn function G_2.

The Horn's function (see [27-29]) are shown as follows:

$$H_2[\alpha,\beta,\gamma,r;s;a,b] = \sum_{f,g=0}^{\infty} \frac{(\alpha)_{f+g}(\beta)_f(\gamma)_g(r)_g}{(s)_f} \cdot \frac{a^f}{f!} \cdot \frac{b^g}{g!},$$

$$(j+1)k = 1 \tag{15}$$

$$H_4[\alpha,\beta;\gamma;r;a,b] = \sum_{f,g=0}^{\infty} \frac{(\alpha)_{2f+g}(\beta)_f}{(\gamma)_f(r)_g} \cdot \frac{a^f}{f!} \cdot \frac{b^g}{g!},$$

$$4j = (k-1)^2 \tag{16}$$

$$H_7[\alpha;\beta,\gamma;a,b] = \sum_{f,g=0}^{\infty} \frac{(\alpha)_{2f+g}}{(\beta)_f(\gamma)_g} \cdot \frac{a^f}{f!} \cdot \frac{b^g}{g!},$$

$$4|a| < 1 \tag{17}$$

the positive quantities j and k are the associated radii of absolute convergence of the double Fourier series $\sum_{f,g=0}^{\infty} \beta_{f,g} a^f b^g$ where $|a| < j$ and $|b| < k$.

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Now, replace a by ak, multiplying by $e^{-k}k^{\alpha-1}\zeta_2\left[\beta;e_1,e_2;bk,ck\right]$ and then integrating term by term with respect to k between the limits 0 to ∞ in (1) we have

$$\Gamma(\alpha)\sum_{g=0}^{\infty}\frac{(\mu)_g(1+\in)_g}{f!.g!}F_E\left[\alpha,\alpha,\alpha,-f-g,\beta,\beta;1+\in,e_1,e_2;a,b,c\right]x^g=$$

$$\int_0^{\infty}e^{-k}k^{\alpha-1}\binom{\in+g}{f}(1-x)^{-\mu}\zeta_2\left[\beta;e_1,e_2;bk,ck\right].\psi_2\left[-f,-\mu;1+\in;ak,\frac{axk}{x-1}\right]dk$$

where F_E and ζ_2 denotes the equation (8) and equation (6) respectively.

Thus by hypothesis,

$$\int_0^{\infty}e^{-k}k^{\alpha-1}\zeta_2\left[\beta,e_1,e_2;ak,bk\right]\psi_2\left[\gamma_1,\gamma_2;r;ak,xk\right]dk=$$

$$\Gamma(\alpha)K_9\left[\alpha,\alpha,\alpha,\alpha;\beta,\beta,\gamma_1,\gamma_2;e_1,e_2,r,r;a,b,c,x\right] \tag{18}$$

where K_9 is Exton's quadruple hypergeometric function which is defined as follows:

$$K_9\left[\alpha,\alpha,\alpha,\alpha;\beta,\beta,\gamma_1,\gamma_2;e_1,e_2,r,r;a,b,c,x\right]=$$

$$\sum_{t,f,g,\ell=0}^{\infty}\frac{(\alpha)_{t+f+g+\ell}(\beta_1)_{t+f}(\gamma_1)_g(\gamma_2)_\ell}{(e_1)_t(e_2)_f(r)_{g+\ell}}\frac{a^t}{t!}.\frac{b^f}{f!}.\frac{c^g}{g!}.\frac{x^\ell}{\ell!} \tag{19}$$

then it takes the form

$$\sum_{g=0}^{\infty}\frac{(\mu)_g}{g!}F_E\left[\alpha,\alpha,\alpha;-f-g,\beta,\beta;1+\in,e_1,e_2;a,b,c\right]x^g=$$

$$(1-x)^{-\mu}K_9\left[\alpha,\alpha,\alpha,\alpha;\beta,\beta,-f-\mu;e_1,e_2,1+\in,1+\in;b,c,a,\frac{xa}{x-1}\right] \tag{20}$$

again replace a by ak,

multiplying by $e^{-k}k^{\alpha-1}{}_1F_1(\beta_3;r_1;\mathrm{bk}).{}_1F_1(\beta_4;r_2;xk)$ and integrating term in (1) then

$$\Gamma(\alpha)K_{13}\left[\alpha,\alpha,\alpha,\alpha;\beta_1,\beta_2,\beta_3,\beta_4;\gamma,\gamma,r_1,r_2;a,b,c,x\right] \tag{21}$$

K_{13} is Exton's quadruple hypergeometric function (see [1]-[7]) defined by

$$K_{13}\left[\alpha,\alpha,\alpha,\alpha;\beta_1,\beta_2,\beta_3,\beta_4;\gamma,\gamma,r_1,r_2;a,b,c,x\right]$$

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$$=\sum_{t,f,g,\ell=0}^{\infty}\frac{(\alpha)_{t+f+g+\ell}(\beta_1)_t(\beta_2)_f(\beta_3)_g(\beta_4)_\ell}{(r_1)_g(r_2)_\ell}.\frac{a^t}{t!}.\frac{b^f}{f!}.\frac{c^g}{g!}.\frac{x^\ell}{\ell!} \tag{22}$$

thus we have

$$\sum_{g=0}^{\infty}\frac{(\mu)_g}{g!}F_A^{(3)}[\alpha;-f-g,\beta_3,\beta_4;1+\in,r_1,r_2;a,b,c]x^g$$

$$=(1-x)^{-\mu}K_{13}\left[\alpha,\alpha,\alpha,\alpha;-f,\mu,\beta_3,\beta_4;1+\in,1+\in,r_1,r_2;a,\frac{ax}{x-1},b,c\right] \tag{23}$$

where $F_A^{(3)}$ is define by equation (13).

Also, replace x by xk, multiplying by $\psi_2[\beta_3,\beta_4;\gamma_2;bk,ck]$ in (1) then

$$\int_0^{\infty}e^{-k}.k^{\alpha-1}\psi_2[\beta_1,\beta_2;\gamma_1;ak,bk].\psi_2[\beta_3,\beta_4;\gamma_2;ck,xk]dk=$$

$$\Gamma(\alpha)K_{12}[\alpha,\alpha,\alpha,\alpha;\beta_1,\beta_2,\beta_3,\beta_4;\gamma_1,\gamma_2,\gamma_2;a,b,c,x] \tag{24}$$

where K_{12} is Exton function (see [1-7]) defined by

$$K_{12}[\alpha,\alpha,\alpha,\alpha;\beta_1,\beta_2,\beta_3,\beta_4;\gamma_1,\gamma_2,\gamma_3;a,b,c,x]=$$

$$\sum_{t,f,g,\ell=0}^{\infty}\frac{(\alpha)_{t+f+g+\ell}(\beta_1)_t(\beta_2)_f(\beta_3)_g(\beta_4)_\ell}{(\gamma_1)_{\ell+\ell}(\gamma_2)_{g+\ell}}.\frac{a^t}{t!}.\frac{b^f}{f!}.\frac{c^g}{g!}.\frac{x^\ell}{\ell!} \tag{25}$$

thus after calculation, we get

$$\sum_{g=0}^{\infty}\frac{(\mu)_g}{g!}F_G[\alpha,\alpha,\alpha,-f-g,\beta_3,\beta_4;1+\in,\gamma_2,\gamma_2;a,b,c]x^g=$$

$$(1-x)^{-\mu}K_{12}\left[\alpha,\alpha,\alpha,\alpha,-f,\mu,\beta_3,\beta_4;1+\in,1+\in;a,\frac{ax}{a-1},b,c\right] \tag{26}$$

where F_G is defined by equation (9).

With the help of the result of author [25], we may easily write

$$\sum_{g=0}^{\infty}\frac{(f+g)!}{(\in+1)_g}\xi_g^{(\in)}(a)\xi_{f+g}^{(\in_1)}(b).x^g=$$

$$(\in_1+1)_f e^b(1-x)^{-\in,-f-1}.\zeta_2\left[\in_1+f+1;\in+1,\in_1+1;\frac{ax}{x-1},\frac{b}{x-1}\right] \tag{27}$$

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where $x \leftarrow (0,1)$.

Replace x by xk and y by yk, multiplying by $e^{-k}.k^{\alpha-1}.\xi_2[\beta_1;\gamma_1,\gamma_2;ck,xk]$ and then integrating, we get

$$\sum_{g=0}^{\infty}\frac{(1+\in_1)_{f+g}}{g!}K_{10}[\alpha,\alpha,\alpha,\alpha;\beta_1,\beta_2,-g,-f-g;\gamma_1,\gamma_2,1+\in,1+\in_1;c,x_1,a,b]x^g =$$

$$\int_0^{\infty} e^{-(1-b)k}.k^{\alpha-1}.(\in_1+1)_g(1-x)^{-\in,-f-1}$$

$$\xi_2[\beta_1;\gamma_1,\gamma_2;ck,x,s].\xi_2\left[\in_1+f+1;\in+1,\in_1+1;\frac{akx}{x-1},\frac{bk}{x-1}\right] \quad (28)$$

Where K_{10} is Exton's function (see [1]-[7]) which is defined as follows:

$$K_{10}[\alpha,\alpha,\alpha,\alpha;\beta,\beta,r;s,w,u,v,a,b,c,x]=$$

$$\sum_{t,f,g,\ell=0}^{\infty}\frac{(\alpha)_{t+f+g+\ell}(\beta)_{t+f}(\gamma)_g(r)_\ell}{(s)_t(w)_f(u)_g(v)_\ell}.\frac{a^t}{t!}.\frac{b^f}{f!}.\frac{c^g}{g!}.\frac{x^\ell}{\ell!} \quad (29)$$

thus

$$\int_0^{\infty} e^{-k}.k^{\alpha-1}.\zeta_2[\beta_1;\gamma_1,\gamma_2;ak,bk].\zeta_2[\beta_2;\gamma_3,\gamma_4;ck,xk]dk =$$

$$\Gamma(\alpha)K_5[\alpha,\alpha,\alpha,\alpha;\beta_1,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3,\gamma_4;\mathrm{a,b,c,x}] \quad (30)$$

where K_5 is Exton function (see [1]-[7]).

In (27), we obtain the following Exton's functions:

$$\sum_{g=0}^{\infty}\frac{(1+\in_1+f)_g}{g!}K_{10}[\alpha,\alpha,\alpha,\alpha;\beta_1,\beta_1;-g-f-g;\gamma_1,\gamma_2,1+\in;1+\in_1;c,x_1,a,b]x^g =$$

$$=\frac{(1-x)^{-\in_1-f-1}}{(1-b)^{\alpha}}K_5\left[\alpha,\alpha,\alpha,\alpha;\beta_1,\beta_1,\in+f+1,\in+f+1;\gamma_1,\gamma_2,\in+1,\in_1+1;\frac{c}{1-b^k},\frac{x_1}{1-b},\right.$$

$$\left.\frac{ax}{(1-b)(x-1)},\frac{b}{(1-b)(x-1)}\right] \quad (31)$$

where Exton function (see [1]-[7]) K_5 is defined as follows:

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$$K_5\left[\alpha,\alpha,\alpha,\alpha;\beta_1,\beta_1,\beta_2,\beta_2;\gamma_1,\gamma_2,\gamma_3,\gamma_4;a,b,c,x\right]=$$
$$\sum_{t,f,g,\ell=0}^{\infty}\frac{(\alpha)_{t+f+g+\ell}(\beta_1)_{t+f}}{(\gamma_1)_t(\gamma_2)_f}\cdot\frac{(\beta_2)_{g+\ell}}{(\gamma_3)_g(\gamma_4)_\ell}\cdot\frac{a^t}{t!}\cdot\frac{b^f}{f!}\cdot\frac{c^g}{g!}\cdot\frac{x^\ell}{\ell!} \quad (32)$$

Case of reducibility:

In equation put $e_1=e_2=\beta$ and with the help of the result of the author [6]:

$$F_E\left[\alpha,\alpha,\alpha,\beta,\gamma,\gamma;r,\gamma,\gamma;a,b,c\right]=$$
$$(1-b-c)^{-\alpha}H_4\left[\alpha,\beta;\gamma,r;\frac{bc}{(1-b-c)^2},\frac{a}{1-b-c}\right]$$

thus we have

$$(1-x)^{\mu}\sum_{g=}^{\infty}\frac{(\mu)_g}{g!}H_4\left[\alpha,-f-g;\beta,1+\in;\frac{bc}{(1-b-c)^2},\frac{a}{1-b-c}\right]=$$
$$(1-b-c)^{\alpha}K_9\left[\alpha,\alpha,\alpha,\alpha;\beta,\beta;-f,\mu;\beta,\beta,1+\in,1+\in;b,c,a,\frac{xa}{x-1}\right] \quad (33)$$

where H_4 is Horn function (see [27]-[29]) defined in equation (16).

In equation (24), replace α by $r_1+\in,r_2$ by β_4 and with the help of result of [6]:

$$K_{13}\left[r+\gamma-1,\alpha+\gamma-1,r+\gamma-1;\beta_1,\beta_2,\beta_3,\beta_4;\gamma,\gamma,r,\beta_4;a,b,c,x\right]=$$
$$(1-a)^{-\beta_1}(1-b)^{-\beta_2}(1-c)^{-\beta_3}(1-x)^{-\beta_4}.D_5\left[\beta_1,\beta_2,\beta_3,r,1-r,1-\gamma;\frac{a}{1-a},\frac{b}{1-b},\frac{c}{1-c},\frac{x}{1-x}\right] \quad (34)$$

thus we have

$$\sum_{g=0}^{\infty}\frac{(\mu)_g}{g!}F_A\left[r_1+\in,-f-g,\beta_3,\beta_4;1+\in,\mathrm{r}_1,\beta_4;a,b,c\right]x^g=(1-a)^f(1+ax-x)^{-\mu}(1-b)^{-\beta_3}$$
$$(1-c)^{-\beta_4}.D_5\left[-f,\mu,\beta_3,r_1,1-r_1,1-r_1,\in_j\frac{a}{1-a},\frac{-ax}{1+ax-x},\frac{b}{1-b},\frac{c}{1-c}\right] \quad (35)$$

where D_5 is Exton's function (see [6]) defined in the following way:

$$D_5\left[\alpha,\beta,\gamma,r,s,w;a,b,c,x\right]=$$
$$\sum_{t,f,g,\ell=0}^{\infty}\frac{(\alpha)_t(\beta)_f(\gamma)_g(r)_\ell(s)_{g+\ell-f-t}}{t!\ f!\ g!\ t!}.a^t.b^f.c^g.x^\ell \quad (36)$$

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In equation (26), replace α by $\in+\gamma_2$ and with the help of the result [6]:

$$F_6\left[\gamma_1+\gamma_2-1,\gamma_1+\gamma_2-1,\gamma_1+\gamma_2-1,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_3;a,b,c\right]=$$

$$(1-a)^{-\beta_1}(1-b)^{-\beta_2}(1-c)^{-\beta_3}$$

$$G_\beta\left[1-\gamma_1,\beta_1,\beta_2,\beta_3;\gamma_2;\frac{a}{a-1},\frac{b}{b-1},\frac{c}{c-1}\right] \qquad (37)$$

For equation (9), we obtain a following function

$$\sum_{g=0}^{\infty}\frac{(\mu)_g}{g!}(1-a)^{f+g}(1-b)^{-\beta_3}(1-c)^{-\beta_4}$$

$$G_\beta\left[-\in,-f-g,\beta_3,\beta_4;\gamma_2;\frac{a}{a-1},\frac{b}{b-1},\frac{c}{c-1}\right]x^g=(1-x)^{-\mu}$$

$$K_{12}\left[\in+\gamma_2,\in+\gamma_2,\in+\gamma_2;-f,\mu,\beta_3,\beta_4;1+\in,1+\in;\gamma_2,\gamma_2;a,\frac{ax}{x-1},b,c\right] \qquad (38)$$

Again in equation (9), arranging proper adjustment of parameters and with the help of result of [6]:

$$K_{12}\left[\gamma_1+\gamma_2-1,\gamma_1+\gamma_2-1,\gamma_1+\gamma_2-1,\gamma_1+\gamma_2-1;\beta_1,\beta_2,\beta_3,\beta_4;a,b,c,x\right]=$$

$$(1-a)^{-\beta_1}(1-b)^{-\beta_2}(1-c)^{-\beta_3}(1-x)^{-\beta_4}$$

$$D_5\left[\beta_1,\beta_2,\beta_3,\beta_4,1-\gamma_1,1-\gamma_2;\frac{a}{1-a},\frac{b}{1-b},\frac{c}{1-c},\frac{x}{1-x}\right] \qquad (39)$$

For Extron function (see [1]-[7]), we obtain a following function:

$$F_G\left[\in+\gamma_2,\in+\gamma_2,\in+\gamma_2,-f-g,\beta_3,\beta_4;1+\in,\gamma_2,\gamma_2;a,b,c\right]x^g=$$

$$(1-a)^f(1+ax-x)^{-\mu}(1-b)^{-\beta_3}(1-c)^{-\beta_4}$$

$$D_5\left[-f,\mu,\beta_3,\beta_4,-\in,1-\gamma_2;\frac{a}{1-a},\frac{-ax}{1+ax-x},\frac{b}{1-b},\frac{c}{1-c}\right] \qquad (40)$$

In equation (26), replace γ_2 by $\alpha,\in$ by $\alpha-1$ and using the help of the result of author [6]:

$$K_{16}\left[\beta_1,\beta_2,\beta_3,\beta_4;\alpha;\frac{ac}{(1-a)(1-c)},\frac{ax}{(1-a)(1-x)},\frac{bc}{(1-b)(1-c)},\frac{bx}{(1-b)(1-x)}\right]=$$

$$(1-a)^{\beta_1}(1-b)^{\beta_2}(1-c)^{\beta_3}(1-x)^{\beta_4}$$

$$K_{12}\left[\alpha,\alpha,\alpha,\alpha;\beta_1,\beta_2,\beta_3,\beta_4;\alpha,\alpha,\alpha,\alpha;a,b,c,x\right] \qquad (41)$$

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thus after calculation, we obtain

$$\sum_{g=0}^{\infty}\frac{(\mu)_g}{g!}F_G\left[\alpha,\alpha,\alpha,-f-g,\beta_3,\beta_4;\alpha,\alpha,\alpha;a,b,c\right]x^g=$$

$$(1-a)^f(1+ax-a)^{-\mu}(1-b)^{-\beta_3}(1-c)^{-\beta_4}$$

$$K_{16}\left[-f,\mu,\beta_3,\beta_4;\alpha;\frac{ab}{(1-a)(1-b)},\frac{ac}{(1-a)(1-c)},\frac{abx(x-1)}{(x-ax-1)(1-b)},\frac{abc(x-1)}{(x-ax-1)(1-c)}\right] \quad (42)$$

where K_{16} is Exton function (see [1]-[7]) defined in the following way:

$$K_{16}\left[\alpha,\beta,\gamma,r;w;a,b,c,x\right]=$$

$$\sum_{t,f,g,\ell=0}^{\infty}\frac{(\alpha)_{t+f}(\beta)_{\ell+g}(\gamma)_{f+\ell}}{(w)_{t+f+g+\ell}}.\frac{a^t}{t!}.\frac{b^f}{f!}.\frac{c^g}{g!}.\frac{x^\ell}{\ell!} \quad (43)$$

CONCLUSION:

Exton (see [1]-[7]) studies some properties of the hypergeometric functions of four variables. The significance of Exton quadruple hypergeometric functions in physical theory becomes more understandable.

In this note, we obtain a set of generating functions of Lurice lla's F_A, Saran's F_E and F_G, Horn's H_4 Exton's K_{10} in terms of quadruple series $K_5,K_9,K_{10},K_{13},K_{16}$ associated with D_5.

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