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THE EUTROPHICATION PROCESS AND ITS EFFECTS ON THE AQUATIC POPULATION - A MATHEMATICAL MODEL

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Abstract

It is largely observed that the growth of algae from industrial waste disposal and chemical runoff from farming fields causes eutrophication of aquatic systems. This process results in an increase of algal blooms and algae. This lowers the water's traceability and the level of dissolved oxygen. The growth of large portion of the aquatic species, including fish, is negatively impacted by an oxygen shortage and a decrease of opacity. With this in view a study is carried out on eutrophication affecting the aquatic species. In this paper, a research investigation on impact of eutrophication process on aquatic species' survival in the water has been developed as a mathematical model based on non-linear ordinary differential equations. The model includes nutrient concentration, growth rate of algae, dissolved oxygen, detritus, and fish population. In order to analyze the formulated model, three equilibrium points were considered to analyze the local stability. The first and second equilibrium points, E_1 and E_2 respectively show unstable state, whereas third equilibrium point E_3 is nontrivial and locally asymptotically stable. Finally the numerical simulation is performed to see the rate of nutrient input affecting different variables that are part of the proposed model.

Keywords: Eutrophication, Discharge of Nutrients, Growth Rate of Algae, Dissolved Oxygen, Detritus Concentration

1. INTRODUCTION:

The main issue affecting the majority of surface water nowadays is eutrophication. It affects aquatic habitats from the Ice cap to the Antarctic, making it among the most obvious examples of human alterations to the biosphere. Nutrient enrichment in the water is due to waste disposal from the industry and agricultural areas. This increases quantity of phytoplankton and algae, lowering the dissolved oxygen and diminishing the clarity of aquatic bodies, as a result of nutrient enrichment produced by many aquatic populations' development rates slow down as a result of oxygen deficiency and the habitat suffers. Eutrophication is a natural ageing process that affects lakes. The slow accumulation of silt and organic waste in the lake causes it. Low nutrient availability and plant productivity characterize a young lake. These oligotrophic lakes

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gradually absorb nutrients from their drainage basin, allowing aquatic life to flourish. The increased biological productivity makes the water murky with phytoplankton in the long run.

The lake becomes eutrophic as the decomposing organic matter depletes oxygen. Algal blooms die and deteriorate, resulting in gloom, odorous clumps of rotting waste, and a decrease in oxygen levels. The amount of the sunlight available to power photosynthetic reactions, as well as the concentration of nutrients required for development, are all elements that influence the rate of production of algae. The amount of light available is proportional to the water's transparency, which is a measure of the concentration of eutrophication.

The use of mathematical modelling in the research and analysis of the impacts of eutrophication on aquatic populations has proven to be quite beneficial. In [1], the researchers suggest integrating the two models to create and study a biological idea of three aquatic species, two of which are competing competition with one another and one of which is a prey-predator. In [2], the researchers put forth mathematical model to investigate the fish population survivability or mortality while taking into the account of impact of both directly and indirectly nutrient recycling under the unfavorable impacts of eutrophication. In [4], a mathematical model is suggested in this research to investigate the impact of declining dissolved oxygen on the coexistence of interactive planktonic ecosystems. In [6], authors provide a model that takes into account interactions between several sea grass species and is based on the norms of plant clonal proliferation. In [7], researchers considered five connected ordinary differential equations are formulated the model. The steady-state dynamics of the model are investigated by using qualitative concept in differential equations.

In [8], the authors have analyzed the topic through a weighted graph with only two cycles; the stability is derived in order to comprehend the behavior of possible cleaners. Numerical simulation is used to validate the model. In [9], mathematical models are developed for various populations to examine the impact of environmental contaminants and rainfall intensity. In [10], the reduction in dissolved oxygen in a water body caused by the discharge of organic waste from residences and industries is investigated using a nonlinear mathematical model. The effect of low amounts of dissolved oxygen on the presence of living organisms is examined in such an aquatic system. In [11], the model analysis demonstrates that the simultaneous impacts of pollution and nutrient enrichment result in a significantly greater fall in DO concentration than when only one factor is found in the body, increasing uncertainty over the existence of DO-dependent species. In [12], the study focuses on a nonlinear model for the high nutrient flow through domestic sewage and the precipitation runoff from farming areas that causes an algae growth in a reservoir. Additionally, it has been shown that the optimum amount of dissolved oxygen decreases and that of detritus grows when the rate of ingestion of cumulative discharge improves. It is crucial to model the dynamic shifts inside the algae species in aquatic reservoirs. Realistic modelling is extremely difficult, though, because of the diversity and highly nonlinear of hydrological factors and associated underlying mechanisms [13]. Under some environmental conditions, naturally existing algae can multiply, leading to the deoxygenation of saltwater or the release of harmful substances (phycotoxins), which can harm both wild and farmed fish and shellfish as well as human consumers [14]. The authors findings

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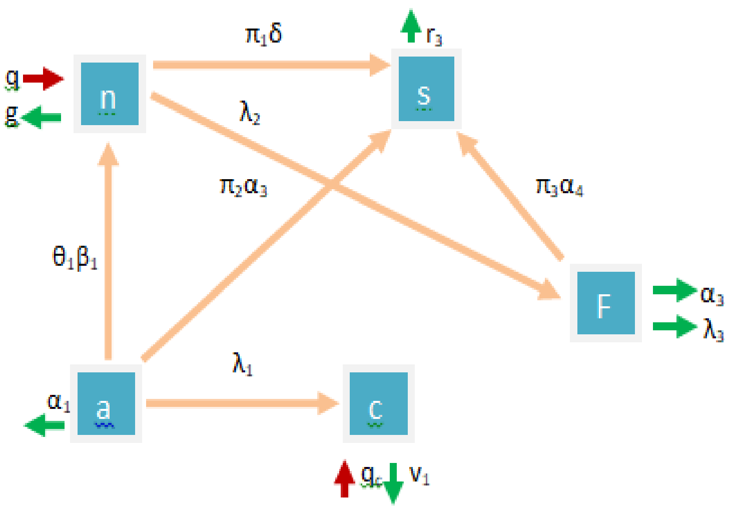
shed light on how fish management and catchment-level restoration strategies might work together to protect and improve water quality [15]. The findings suggest that approaches for nutrient reduction, rather than planktonic bloom monitoring, should then be developed to reach beyond unique conditions and across a river-estuary-ocean spectrum, taking into account the possibility that extreme nutrients and fresh-water phytoplankton blooms could be transmitted to ground water shores, shorelines, and occasionally even waterways that are susceptible to poor quality of water[17].

2. MATHEMATICAL MODEL:

In this model a(t) is the Growth rate of algae is directly proportion to the enrichment of nutrients(na) along with the oxygen deficit. n(t) is the increased growth of algae in an area with low oxygen due to nutrient concentration is also taken into account when calculating the total rate of increase of nutrients from detritus. c(t) is the rate of natural depletion of concentration C is considered to be v1, and growth of dissolved oxygen from various sources is supposed to be qc. Algae are thought to degrade at a pace that is proportional to both a and a2. Detritus is created when algae die and descend to the lake's bottom, therefore its growth rate must be proportionate to a. s(t) is a naturally occurring depletion of detritus brought on by the decaying process brought upon with bacteria or fungus activity in the lake. The density of debris in the lake affects how quickly nutrients grow because detritus is broken down by microbes and produces nutrients. F(t) rate of fish is Decay rate of fish by carrying capacity of fish. The mathematical model is developed using the relevant presumptions and parameters that were previously given. Figure 1 shows the connection network of a model.

Figure 1: Eutrophication model diagram



The five state variables listed above are included in the mathematical model of the governing system, which is organised as follows:

The Seybold REPORT
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$$\frac{dn}{dt} = q - \pi_1 \delta s + \alpha_2 n - g n^2 - - - - - - (1)$$
$$\frac{da}{dt} = \theta_1 \beta_1 n a - v_1 a^2 + \alpha_1 a - - - - - - (2)$$
$$\frac{dc}{dt} = q_c + v_1 c - \lambda_1 a - - - - - - - - - (3)$$
$$\frac{ds}{dt} = \pi_2 \alpha_3 a + \pi_3 \alpha_4 F - v_3 s - - - - - - (4)$$
$$\frac{dF}{dt} = \lambda_2 n F + \alpha_3 F - \lambda_3 F^2 - - - - - - -(5)$$

With initial conditions

$a(0)=a_0>0$, $n(0)=n_0>0$, $c(0)=C_0>0$, $s(0)=s_0>0$, $f(0)=f_0>0$
$\beta_1, \beta_2, \theta_1, \theta_2, \theta_2, \delta_1, \delta$ are proportionality constant which are positive.
α_i`s are depletion rate coefficients.
$\lambda_1, \beta_1, \alpha_1, \alpha_2, \delta, \alpha_3, \alpha_4$ are proportionality constants which are positive.
v_1, v_3 are coefficients corresponding to crowding.
π_1, π_2, π_3 are functional proportionality constants.
q is the cumulative rate of nutrient discharge into the exterior water body.
q_c is the rate of DO growth according to different sources (assumed as constant).

3. BOUNDEDNESS AND DYNAMICAL BEHAVIOUR:

The boundedness of the solutions of the model (1) – (5) is given by the following lemma

Lemma1: The set $\Omega = \{(n,a,s,F) \in R_4^+ : 0 \le n+a+s+F \le \frac{q}{\delta_m}, c \le \frac{q_c \delta_m - \lambda_1 a}{\delta_m}$ is a region for all positive solution where $\delta_m = \min\{\alpha_1, \alpha_2, \alpha_3, v_1, v_2\}$

Proof: Using the first four systems (1) equations that we find,

$$\frac{d}{dt}(n+a+F+s) = \frac{dn}{dt} + \frac{da}{dt} + \frac{dF}{dt} + \frac{ds}{dt}$$

$$\le q + \alpha_2 n - v_1 a + \alpha_3 F - v_3 s \text{ ----------------- } (6)$$

$$\le q + \delta_m (n - a + F - s)$$

Here B=n-a+F-s

$$\frac{dB}{dt} - \delta_m B \le q$$

Solve the equation (3) using integration factors

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$$B \le \frac{q}{\delta_m} + ke^{\delta_m}$$

As $t \to \infty$ we have

$$0 \le B \le \frac{q}{\delta_m}$$

$$0 \le n - a + F - s \le \frac{q}{\delta_m}$$

Replace the maximum values from equation with the fourth system's equation (4)

$$\frac{dc}{dt} \le q_c + v_2 c - \lambda_1 \frac{a}{\delta_m}$$

$$\frac{dc}{dt} - v_2 c \le \frac{q_c \delta_m - \lambda_1 a}{\delta_m}$$

Integrating equation (5) we get,

$$0 \le c \le \frac{q_c \delta_m - \lambda_1 a}{\delta_m} + ke^{tv_2}$$

As $t \to \infty$, we have

$$0 \le c \le \frac{q_c \delta_m - \lambda_1 a}{\delta_m}$$

This completes the proof of lemma.

4. ANALYSIS OF EQUILIBRIA

Let us now discuss about the stability of interior and boundary equilibrium points. The model's non-negative equilibrium points are as follows.

Case 1: $E_1[n \neq 0, a = 0, c = 0, s = 0, F \neq 0]$ always exists

$$n = \frac{\alpha_2 \pm \sqrt{\alpha_2^2 - 4gq}}{2g}$$

$$F = \frac{1}{\lambda_3}\left[\lambda_2\left(\frac{\alpha_2 \pm \sqrt{\alpha_2^2 - 4gq}}{2g}\right) + \alpha_3\right]$$

This model 1 equilibrium shows how when algae are present in the system, the detritus density will decrease to zero at equilibrium. It's also important to note that detritus is produced when algae die and that fish do not reduce the amount of nutrients in the system.

Case 2: $E_2[n \neq 0, a = 0, c \neq 0, s \neq 0, F \neq 0]$

The Seybold Report
ISSN 1533-9211
DOI 10.5281/zenodo.7451525

$$n = \frac{\alpha_2 \pm \sqrt{\alpha_2^2 - 4gq}}{2g}$$

$$F = \frac{1}{\lambda_3}\left[\lambda_2\left(\frac{\alpha_2 \pm \sqrt{\alpha_2^2 - 4gq}}{2g}\right) + \alpha_3\right]$$

$$c = \frac{q_c}{v_1}$$

To check the effect of nutrients, concentration of oxygen, density of detritus as well as fish population on algae(phytoplankton) tend to zero.

Case 3: $E_3[n \neq 0, a \neq 0, c \neq 0, s \neq 0, F \neq 0]$

$$n = \frac{\alpha_2 \pm \sqrt{\alpha_2^2 - 4gq}}{2g}$$

$$F = \frac{1}{\lambda_3}[\lambda_2 n + \alpha_3]$$

$$a = \frac{\theta_1 \beta_1 n}{v_1} + \frac{\alpha_1}{v_1}$$

$$c = \frac{q_c + \lambda_1 a}{v_1}$$

5. LOCAL STABILITY ANALYSIS:

To build the variational matrix to determine the behaviour of the equilibrium points' local stability.

$$J = \begin{bmatrix} \alpha_2 - 2gn & 0 & 0 & -\pi_1\delta & \lambda_2 n \\ \theta_1\beta_1 a & \theta_1\beta_1 n - 2v_1 a + \alpha_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & -v_1 & 0 & 0 \\ 0 & \pi_2\alpha_3 & 0 & -v_3 & \pi_3\alpha_4 \\ \lambda_2 n & 0 & 0 & 0 & \lambda_2 n + \alpha_3 - 2\lambda_3 F \end{bmatrix}$$

At $E_1[n \neq 0, a = 0, c = 0, s = 0, F \neq 0]$

$$J_1 = \begin{bmatrix} \alpha_2 - 2gn & 0 & 0 & -\pi_1\delta & 0 \\ 0 & \theta_1\beta_1 n + \alpha_1 & 0 & \pi_2\alpha_3 & 0 \\ 0 & \lambda_1 & -v_1 & 0 & 0 \\ 0 & \pi_2\alpha_3 & 0 & -v_3 & \pi_3\alpha_4 \\ \lambda_2 F & 0 & 0 & 0 & \lambda_2 n + \alpha_3 - 2\lambda_3 F \end{bmatrix}$$

ISSN 1533-9211
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$|J_1 - \alpha I| = 0$

$\Rightarrow b^2 - 4ac \geq b^2$

=> Unstable

At $E_2[n \neq 0, a = 0, c \neq 0, s \neq 0, F \neq 0]$

$|J_2 - \alpha I| = 0$

⇨ $b^2 - 4ac \geq b^2$

⇨ Unstable

At $E_3[n \neq 0, a \neq 0, c \neq 0, s \neq 0, F \neq 0]$

$$J_3 = \begin{bmatrix} \alpha_2 - 2gn & 0 & 0 & -\pi_1\delta & \lambda_2 n \\ \theta_1\beta_1 a & \theta_1\beta_1 n - 2v_1 a + \alpha_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & -v_1 & 0 & 0 \\ 0 & \pi_2\alpha_3 & 0 & -v_3 & \pi_3\alpha_4 \\ \lambda_2 n & 0 & 0 & 0 & \lambda_2 n + \alpha_3 - 2\lambda_3 F \end{bmatrix}$$

$$b^2 - 4ac \geq 0$$

This shows that J_3 is locally asymptotically stable and exists.

6. NUMERICAL SIMULATION

Let's perform some numerical calculations, choosing the following values for the model's component values, to test the validity of this analysis about the stability of E_3 of model (1)

$q = 5.0, \pi_1 = 0.1, \delta = 1.0, \alpha_1 = 0.5, g = 1.0, \theta_1 = 1.0, \alpha_2 = 0.5, v_1 = 2.0, \beta_1 = 1.0$

$q_c = 10.0, v_3 = 2.0, \lambda_1 = 0.25, \pi_2 = 0.9, \alpha_3 = 0.5, \alpha_4 = 2.0, \lambda_2 = 0.4, \lambda_3 = 0.122,$

$\alpha_3 = 0.02$ ----------- (6.1)

The prerequisites for the existence of internal equilibrium are noted to E_3

$(n^*, a^*, c^*, S^*, F^*)$ satisfy the aforementioned set of requirements, and E_3 is given by $n^* = 3.422$, $a^* = 1.961$, $c^* = 4.7487$, $s^* = 3.8877$, $F^* = 15.3180$

The Jacobian matrix eigenvalues for this equilibrium, E_3, are as follows:

$-2.000 + 0.0000i$

$-6.5753 + 0.0000i$,

$-3.8698 + 0.0000i$,

$-1.8142 + 0.9740i$,

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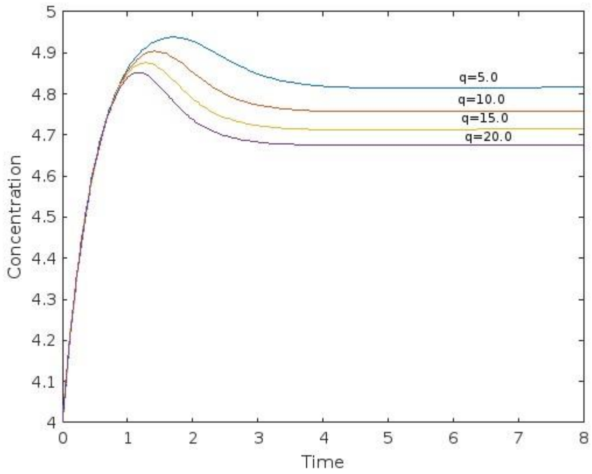
$-1.8142 + 0.9740i$

In this case, there are three real eigenvalues, two complex eigenvalues, and all of the eigenvalues are negative or have a negative real portion. E_3 is therefore locally asymptotically stable.

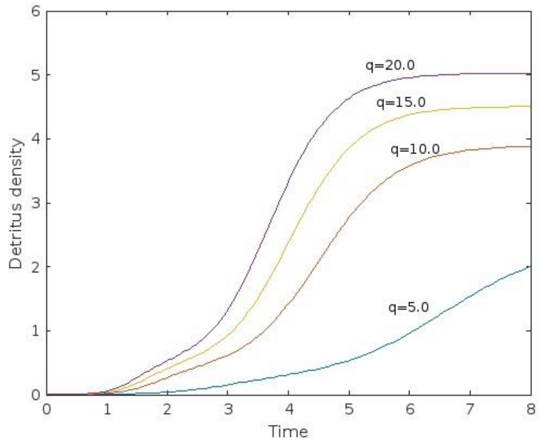
By maintaining the other parameters constant (as given in Figures 2–5), we can see the impact of q on S, C and F.

Figure 2. Variation of concentration of nutrients(C) varying cumulative rates of nutrient (q) availability with respect to time



In figure 2, By maintaining the other parameters constant as stated in (6.1), it is noticed that the influence of the amount of input of nutrient q on the concentration of nutrients (c) causes the concentration of nutrients to increase.

Figure 3. Variation of Detritus density (S) varying cumulative rates of nutrient(q) availability with respect to time



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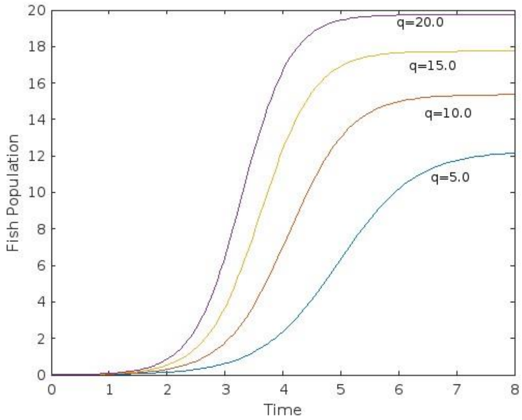
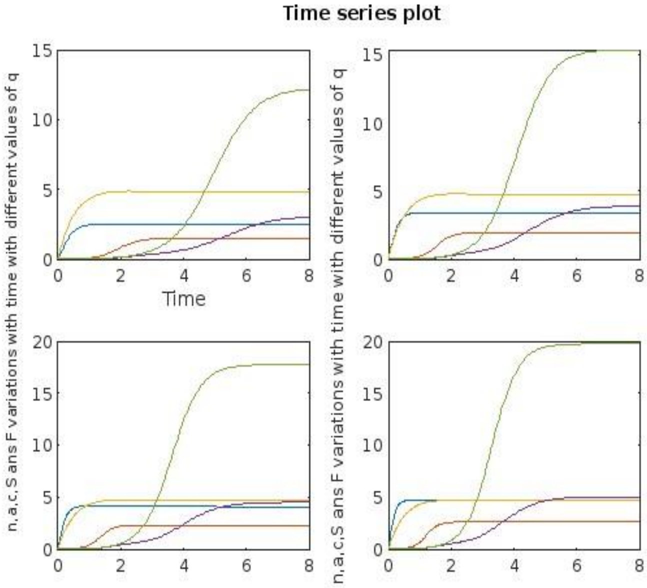
In figure 3, by holding all other parameters constant as stated in (6.1), it can be seen that the influence of the rate of input of nutrient q on the density of detritus (s) results in a declining, or tending to zero, density of detritus.

Figure 4. Variation of Fish population (F) depending on the cumulative rate of nutrient (q) release throughout time



In figure4, holding other parameters constant as stated in (6.1), it is noticed that the influence of the amount of input of nutrient q on Fish population (F) results in an increase in Fish population. While the concentration of dissolved oxygen reaches its maximum, the rate of nutrient introduction (q = 0), the density of detritus, and the density of algae all tend to zero. This result is inevitable since the nutrients produced by the detritus won't be sufficient to sustain the expansion of the fish, algal, and detritus populations.

Figure 5. Time series graph for N-F variations with different values of q



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The inner equilibrium point E_3 is demonstrated to be stable by the numerical simulations. Figure 5 time series graphs for model includes nutrient concentration, growth rate of algae, dissolved oxygen, detritus and fish population is illustrated the findings to demonstrate the density of eutrophication rate increases and the density of fish population declines with varying amounts of nutrients (q).

7. CONCLUSION

This research proposes and investigates a non-linear mathematical model to study the eutrophication of water due to the overpopulation of fish, algae, and other biological species caused by an excessive supply of nutrients from water runoff, agricultural fields, industries, households and other sources. According to the observations, eutrophication process begins when the number of algae and other biological species in a body of water increase as the amount of nutrients in the water body goes up.

Additionally, it has been studied that as detritus density rises, dissolved oxygen concentration falls i.e. the dissolved oxygen content is unaffected as oxygen produced by floating algae's photosynthesis enters the atmosphere. It has been observed through simulation study that if the rate of outside nutrient input is large, the content of dissolved oxygen in a water body may become insignificant.

The model containing three equilibrium points i.e., $E_1(n,0,0,0,F)$, $E_2(n,0,c,s,F)$ and $E_3(n^*, a^*, c^*, s^*, F^*)$ for the model. The first and second equilibrium points E_1 and E_2 respectively show an unstable state, whereas non trivial equilibrium point E_3 is locally as well as asymptotically stable. The theoretical values and numerical simulated value are validated. This implies that the rise in fish population, algal, and detritus populations will not be supported more by nutrients that produces detritus. As the current study focuses on aquatic population (specifically fish population) in the presence of eutrophication, however, further studies can be carried out to by considering different aquatic species e.g. zooplankton, phytoplankton, types of fish population and so on .

References

1. Agmour, I., Achtaich, N., & Foutayeni, Y. E. "Stability analysis of a competing fish population's model with the presence of a predator." International Journal of Nonlinear Science, 2018: 26(2), 108-121.
2. Brown, A. R., Lilley, M., Shutler, J., Lowe, C., Artioli, Y., Torres, R., & Tyler, C. R. "Assessing risks and mitigating impacts of harmful algal blooms on mariculture and marine fisheries." Reviews in Aquaculture, 2020: 12(3), 1663-1688. https://doi.org/10.1111/raq.12403
3. Chaturvedi, A., & Misra, O. P. "Modelling Effects of eutrophication on the survival of fish population incorporating nutrient recycling." J. of Int. Acad. Of Phy. Sci, 2010: 14, 487-500.
4. Dunca, A. M. "Water pollution and water quality assessment of major transboundary rivers from Banat." Journal of Chemistry, 2018. https://doi.org/10.1155/2018/9073763
5. Khare, S., Kumar, S., & Singh, C. "Modelling effect of the depleting dissolved oxygen on the existence of interacting planktonic population." Elixir Appl. Math, 2013: 55, 12739-12742.

ISSN 1533-9211

DOI 10.5281/zenodo.7451525

6. Khare, S., Misra, O. P., & Dhar, J. "Effect of soil pollutant on the plant–herbivore interacting system incorporating nutrient cycling: A mathematical model." J. Scientific Research, 2009: 53, 163-174.
7. Llabrés, E., Mayol, E., Marbà, N., & Sintes, T. "A mathematical model for inter-specific interactions in seagrasses." Oikos, 2022: e09296. https://doi.org/10.1111/oik.09296
8. Lou, I., Xie, Z., Ung, W. K., & Mok, K. M. "Integrating support vector regression with particle swarm optimization for numerical modeling for algal blooms of freshwater." In Advances in Monitoring and Modelling Algal Blooms in Freshwater Reservoirs-Springer, Dordrecht. 2017: 125-141. DOI: 10.1007/978-94-024-0933-8_8
9. Mao, Z., Gu, X., Cao, Y., Zhang, M., Zeng, Q., Chen, H., & Jeppesen, E. "The role of top-down and bottom-up control for phytoplankton in a subtropical shallow eutrophic lake: evidence based on long-term monitoring and modeling. . ." Ecosystems, 2020: 23(7), 1449-1463.
10. Misra, A. K. "Mathematical modeling and analysis of eutrophication of water bodies caused by nutrients." Nonlinear Analysis: Modelling and Control, 2007: 12(4), 511-524. doi: 10.15388/NA.2007.12.4.14683.
11. Shah, N. H., Satia, M. H., & Thakkar, F. A. "Effect of Environmental Pollutants on Rain due to Stakeholders." In Mathematics in Engineering Sciences: Novel Theories, Technologies, and Applications, 2019: 301-327.
12. Shah, N. H., Satia, M. H., & Yeolekar, B. M. "Stability of 'GO-CLEAN' model through graphs." Journal of Computer and Mathematical Sciences, 2018: 9(2), 79-93.
13. Shukla, J. B., Misra, A. K., & Chandra, P. "Mathematical modeling and analysis of the depletion of dissolved oxygen in eutrophied water bodies affected by organic pollutants." Nonlinear Analysis: Real World Applications, 2008: 9(5), 1851-1865. https://doi.org/10.1016/j.nonrwa.2007.05.016
14. Shukla, J. B., Misra, A. K., & Chandra, P. "Mathematical modeling of the survival of a biological species in polluted water bodies." Differential Equations and Dynamical Systems, 2007: 15(3/4), 209-230.
15. Shukla, J. B., Misra, A. K., & Chandra, P. "Modeling and analysis of the algal bloom in a lake caused by discharge of nutrients. . ." Applied mathematics and computation, 2008: 196(2), 782-790.
16. Smith, V. H., & Schindler, D. W. (2009). Eutrophication science: where do we go from here? Trends in Ecology & Evolution, 24(4), 201-207. doi:10.1016/j.tree.2008.11.009
17. Wang, J., & Zhang, Z. "Phytoplankton, dissolved oxygen and nutrient patterns along a eutrophic river-estuary continuum: Observation and modeling." Journal of environmental management, 2020: 261, 110233.