The Seybold Report

ISSN 1533-9211

DOI 10.17605/OSF.IO/PFZTR

GENERATING PYTHAGOREAN TRIPLES

EDLYN R. NARAG, Ph.D.

Cagayan State University, Andrews Campus, Tuguegarao City, Cagayan. Email: edlynrnarag@gmail.com

MARIA DIVINA B. TAPIRU, MST, MOM

Cagayan State University, Carig Campus, Tuguegarao City, Cagayan. Email: ma.divinatapiru1602@gmail.com

Abstract

This study investigates which numbers can be part of triples such as (3, 4, 5) and (5, 12, 13) right-angled triangles with integer sides. It is limited in generating Pythagorean triples (a, b, c) with no common factor such that a right triangle exists with legs a, b and hypotenuse c and satisfies the equation$a^2 + b^2 = c^2$. It sought to answer the questions: (1) what is a Pythagorean Triple? (2) Prove the general formula in generating Pythagorean triples. It made use of the exploratory approach through investigation and giving proof. The highlights of the study are as follows: (1) The Pythagorean triples was derived from the Pythagorean Theorem which states that "the square of the hypotenuse of a right triangle is equal to the sum of the squares on the two legs", (2) The Pythagorean Triple is a triple of positive integers a, b, c such that $a^2 + b^2 = c^2$. (3) For any choice of integers m and n where n < m, we can get a Pythagorean triple by setting values for a, b and c to show that $(m^2 + n^2)^2 = (m^2 - n^2)^2 + (2mn)^2$. (4) If restrictions on m and n that one of them must be even and the other is odd then, the values of $(m^2 + n^2)$, $(m^2 - n^2)$ and (2mn) have no common factor. It is recommended that teachers should possess the high level of mathematical intellect, integrity and always update themselves in the new trends to cope with the fast-changing world of mathematics.

Keywords: Pythagorean Theorem, Pythagorean Triples

INTRODUCTION

Mathematics Education in the Philippines is faced with the problem of improving the quality of instruction. This deterioration of its quality of learning among learners has been the concern of the whole system of education. The educational system should create an effective and functional school which anticipates the enormous perspective of global change from the context of relevant and responsive educators. In relation to this, Pythagoras, a Greek philosopher and mathematician, was not exaggerating when he claimed that mathematics is the basis of all sciences. The Chemist, Physicist, the Astronomer, the Engineers, Doctors and other fields of specialization could get along without mathematics, because it is encountered every day on our daily undertakings. The banker, insurance man, the accountant, the skilled workers, must know certain branches and concepts of math. Since, mathematics is the queen of science, it is therefore one proficiency which is a pre-requisite to mastering scientific and technological pursuit. Therefore, mathematics contributes directly or indirectly to most phases of modern civilization because it is necessary for development.

The study about the Pythagorean Theorem is used in all aspects of mathematics, particularly in Geometry and Trigonometry. However, the algebra associated with the generation of the triples that satisfy the theorem is often overlooked.

ISSN 1533-9211
DOI 10.17605/OSF.IO/PFZTR

Pythagoras' Theorem is perhaps the best-known result in the whole of mathematics and yet many things remain unknown or perhaps just "unstudied" about the consequences of this simple theorem. In this study, we investigate which numbers can be part of triples such as (3, 4, 5) and (5, 12, 13) – right angled triangles with integer sides.

The Pythagorean triple is an example of a problem in number theory, which is the study of the properties of whole numbers. The good thing about number theory is that students can readily relate to it, since we are all more comfortable with whole numbers than fractions and decimals. The unfortunate thing about number theory is that it has fewer practical applications than other branches of mathematics.

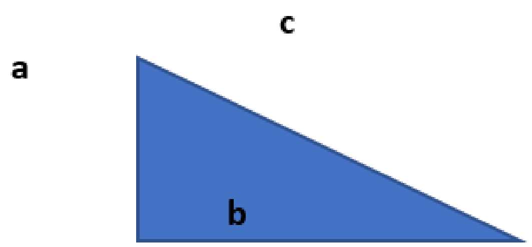
To make mathematics education more responsive and relevant to change taking place in our modern society especially our educational institution, teaching staff must have to examine critically the content, methods and techniques that they employ in teaching mathematics. It should meet the demands and needs of learners in order to facilitate maximum learning to the students. Current education reforms support challenging learning standards and school accountability. In mathematics education, the emphasis is on the development of conceptual understanding and reasoning over memorization and rote learning.

Hence, Educators in Cagayan State University are working so hard in the analysis of the total learning situation in order to improve quality education for this 21st millennium.

REVIEW OF LITERATURE

Pythagoras, (550-500 B.C.) was born on the Aegean Island of Samos. Pythagoras has been called one of the founding fathers of the science of mathematics. His most famous rule or theorem is what he discovered about right angled triangles.

An early number theory is the Pythagorean problem. In a right-angled triangle, the lengths of the sides satisfy the Pythagorean relation $a^2 + b^2 = c^2$, where c is the length of the hypotenuse, the side opposite the right angle.



This equation make it possible to compute the length of one side of right triangle when you know the other two.

Generating a few triples is relatively easy, especially for small values of R and S. For more than few values of R and S, However, the calculations become a chore.

ISSN 1533-9211

DOI 10.17605/OSF.IO/PFZTR

In the number theory trivia of Spencer (1995). The following formulas may be used to generate Pythagorean triples.

1. If m is any odd integer, then the three numbers m, $(m^2 - 1)/2$ and $(m^2 + 1)/2$ yield a Pythagorean triple.
2. If m is any integer, then the three numbers 2m, $m^2 – 1$ and $m^2 + 1$, yield a Pythagorean triple.
3. If x and y are integers and if $a = x^2 – y^2$, $b = 2xy$, $c = x^2 + y^2$, then a, b and c are integers such that $a^2 + b^2 = c^2$.

In the book of Spencer, Donald (1995), the microcomputer can be used to eliminate the drudgery of calculation. He emphasized in his book a program which produces Pythagorean triples. The program produces primitive triples for $0 < S < R < 15$. Modifying lines 200 and 210 for R and S greater than 15 and 14, respectively will give other triples.

Sums of two squares is obtained if each pair of positive integers, u,v with $u > v$, defining $x = 2uv$, $y = u^2 – v^2$, $z = u^2 + v^2$ produces a Pythagorean triple (x, y, z). (Eynden, 2006).

Benett, Dan (1995), he stressed in his book the famous right triangle theorem ($c^2 = a^2 + b^2$) which is named after Pythagoras. He believed that all of nature could be explained by numbers. In fact their motto stated “All things are numbers”. They especially valued the numbers 1, 2, 3, and 4 which they called the tetractys. The Pythagorean oath was “I swear in the name of the tetractys, which has been bestowed on our soul”. They saw fairness in many things, including the four geometric elements (point, line, surface and solid) and the four elements (earth, air, fire and water).

According to Bogomolny (2010), three integers a, b, and c that satisfy $a^2 + b^2 = c^2$ are called Pythagorean Triples. There are infinitely many such numbers and there also exists a way to generate all the triples. Let n and m be integers, $n > m$. Then define $b = 2nm$, $a = n^2 – m^2$, $c = n^2 + m^2$. The three number a, b and c always form a Pythagorean triple. The proof is simple:

$$\begin{aligned}(n^2 – m^2)^2 + (2mn)^2 &= n^4 – 2n^2m^2 + m^4 + 4n^2m^2 \\ &= n^4 + 2n^2m^2 + m^4 \\ &= (n^2 + m^2)^2.\end{aligned}$$

The formulas were known to Euclid and used by Diophantus to obtain Pythagorean triples with special properties. However, he never raised the question whether in this way one can obtain all possible triples.

The Patterns in Pythagorean Triples by Evans, Chris. Mathematical Gazette 75 (1995). He demonstrated patterns of Pythagorean triples which started with two sides of the triangle that form the right-angle, its legs and use the letters a and b. The hypotenuse is the longest side opposite the right-angle and we will often use h for it. The two legs and the hypotenuse are the three sides of the triangle, triple or triad a, b and h.

ISSN 1533-9211
DOI 10.17605/OSF.IO/PFZTR

The series of lengths of the hypotenuse of primitive Pythagorean triangles begins 5, 13, 17, 25, 29, 37, 41, etc. in Sloane's Online Encyclopedia of Integer Sequences, it will contain 65 twice – the smallest number that can be the hypotenuse of more than one primitive Pythagorean triangle. The series of numbers that are the hypotenuse of more than one primitive Pythagorean triangle is 65, 85, 145, 185, 205, 221, 265, 305, etc. There are lots of patterns in the list of Pythagorean Triples. To start off his investigations here are a few

The first and simplest Pythagorean triangle is the 3, 4, 5 triangle. Also near the top of the list is 5, 12, and 13. In both of these, the longest side and the hypotenuse are consecutive integers. The list below tells us there are other more Pythagorean triangles. Here are the following:

3	4	5
5	12	13
7	24	25
9	40	41
11	60	61

When Chris Evans was a student of Diss High School, he found the following pattern where the fractions give the two sides and the hypotenuse is the numerator +1.

$$1\frac{1}{3} = \frac{4}{3} \quad (3, 4, 5)$$

$$2\frac{2}{5} = \frac{12}{5} \quad (5, 12, 13)$$

$$3\frac{3}{7} = \frac{24}{7} \quad (7, 24, 25)$$

$$4\frac{4}{9} = \frac{40}{9} \quad (9, 40, 41)$$

$$5\frac{5}{11} = \frac{60}{11} \quad (11, 60, 61)$$

According to **Weisstein (2008)** in his research work on Twin Pythagorean triple states "a Pythagorean triple is a triple of positive integers a, b and c such that a right triangle exist with legs a, b and hypotenuse c". By the Pythagorean Theorem this is equivalent to finding positive integers a, b and c satisfying the equation $a^2 + b^2 = c^2$. The smallest and best known Pythagorean triple is (a, b, c) = (3, 4, and 5).

Based on the study of **Overmars, A, et.al. (2019)** entitled "A New Approach to generate all Pythagorean Triples", it presented an approach that parametrise the Pythagorean Triples which generates all of the triples in a unique way without repetitions. They also explored the relation of this new parametrisation with the Pythagorean family of odd triples and the Platonic family of even triples.

On the other hand, **Amato, R., (2017)** presented in his paper entitled "A characterization of Pythagorean Triples", an analytic result which characterizes the Pythagorean triples via a

The Seybold REPORT

ISSN 1533-9211

DOI 10.17605/OSF.IO/PFZTR

cathetus. This is an easy method to find the Pythagorean triples $x, y, z \in N$, where x is a predetermined integer, which means finding all right triangles whose sides have integer measures and one cathetus is predetermined.

STATEMENT OF THE PROBLEM

This study aimed to investigate the general formula in generating Pythagorean triples.

Specifically, it sought to answer the following questions:

1. What is a Pythagorean triple?
2. Show and prove the general formula in generating Pythagorean triples.

SIGNIFICANCE OF THE STUDY

The findings of the study will contribute to the quality of classroom instruction in Mathematics. It will benefit the following sectors:

The Institution: The school will be benefited on the content of the study presented through the grantee of the Program sponsored by CHED. They served as a partner in promoting good values, developing skills and nurturing the mind of the students.

CHED: The Commission on Higher Education will have a basis in offering such scholarship to deserving faculty in improving mathematics instruction in the Philippine setting.

Students: The result of this study will be beneficial to the students who are recipients of knowledge, skills and values. Furthermore, they will learn more effectively on the concern of this study.

Teachers: The outcomes of this study will help mathematics teachers both in high school and college attain high level of mathematical intellect which could be a tool in the delivery of mathematics instructions related to Pythagorean triples.

Future Researchers: The finding of this study will serve as a guide for future researchers to investigate more about Pythagorean triples.

Researchers: The result of this study will be of great help to the researchers in teaching effectively and enjoyably this particular topic in Mathematics.

SCOPE AND LIMITATION OF THE STUDY

This study is limited in generating Pythagorean triples (a, b, c) with no common factor such that a right triangle exists with legs a, b and hypotenuse c and satisfies the equation $a^2 + b^2 = c^2$.

In the discussion, it is shown that we can generate all such triples by adding further restrictions on m and n and that one of them must be positive even integer and the other is positive odd integer and that m and n have no common factor.

ISSN 1533-9211

DOI 10.17605/OSF.IO/PFZTR

The study of Pythagorean triples can be purely with numbers. In this connection, it is likely to make the problem less abstract to the students in the curriculum of high school and college.

METHODOLOGY

Research Design

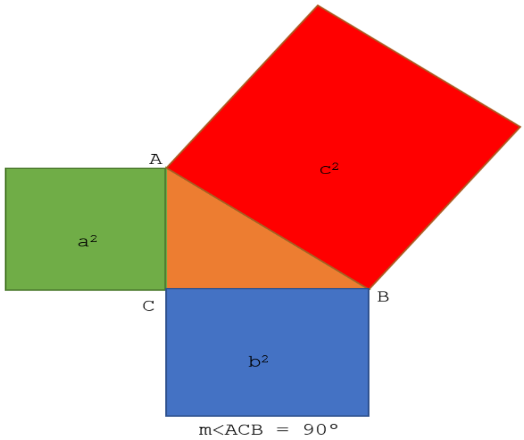
The research method that was employed was the exploratory approach since the chief purpose of the study is to investigate the general formula in generating Pythagorean triples with no common factor.

Research Locale

Preliminary Notions

Definition 1. The Pythagorean Theorem states that "If a triangle has one angle which is a right-angle (i.e. 90º) then there is a special relationship between the lengths of its three sides: If the longest side (called the hypotenuse) is c and the other two sides (next to the right angle) are called a and b, then $a^2 + b^2 = c^2$."

Illustration. This theorem is talking about the area of the squares that are built on each side of the right triangle.



Accordingly, we obtain the following areas for the squares, where the green and blue squares are on the legs of the right triangle and the red square is on the hypotenuse.

Area of the green square is a^2

Area of the blue square is b^2

Area of the red square is c^2

Definition 2: A Pythagorean Triple is a triple of positive integers a, b and c such that a right triangle exists with legs a, b and hypotenuse c and satisfies the equation

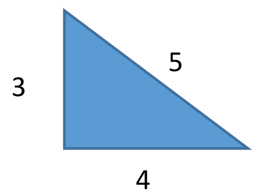
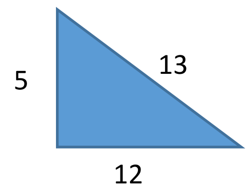
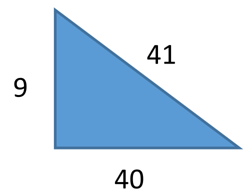
$$a^2 + b^2 = c^2$$

The Seybold Report
ISSN 1533-9211
DOI 10.17605/OSF.IO/PFZTR

Illustration: By the Pythagorean Theorem, this is equivalent to finding positive integers a, b and c satisfying $a^2 + b^2 = c^2$.

The smallest example is a = 3, b = 4 and c = 5. You can check that $3^2 + 4^2 = 9 + 16 = 25 = 5^2$. Sometimes we use the notation (a, b, c) to denote such a triple. Notice that the greatest common divisor of the three numbers 3, 4 and 5 is 1.

Here are some examples:

		
3,4,5 Triangle	5,12,13 Triangle	9,40,41 Triangle
$3^2 + 4^2 = 5^2$	$5^2 + 12^2 = 13^2$	$9^2 + 40^2 = 41^2$

DATA ANALYSIS

Main Body

Theorem: If a, b and c are integers satisfying

$$a^2 + b^2 = c^2,$$

Where a, b and c have no common factor then:

1. c is an odd number and
2. One of a and b is odd and the other even.

Proof: Since the sum and difference of two even numbers is even, if two of a, b and c were even then the other would be also, which is not allowed since the numbers would then have a common factor of two. If none of a, b and c were even then a, b and c would all be odd, which is impossible since the sum or difference of two odd numbers is an even number. Therefore, exactly one of a, b and c is even.

Suppose that c is the even number. Then c has a factor of 2 and c^2 has a factor of 4 that is, c^2 is divisible by 4. If we choose two odd numbers, say 1 and 3 and add their squares which are 10, we will find that the sum is not divisible by 4. Since a and b are both odd they have the form a = 2x + 1 and b = 2y + 1. Then $a^2 + b^2 = (2x + 1)^2 + (2y + 1)^2 = 4(x^2 + y^2 + x + y) + 2$, which has a remainder of 2 when divided by 4 and so cannot be equal to c^2, which is exactly divisible by 4. Therefore, the assumption that c is even is incorrect, since we showed that either a or b must be even and c must always be odd.

The Seybold Report
ISSN 1533-9211
DOI 10.17605/OSF.IO/PFZTR

By factorization:

From $c^2 = a^2 + b^2$, we get

$c^2 – a^2 = b^2$,

$(c + a) (c - a) = b^2$.

If we can choose a and c so that (c + a) (c - a) is a perfect square then we generate a triple.

b^2 is divisible by 2 since we are assuming that b is even and both (c + a) and (c - a) are even since c and a are both odd

$(c + a)/2 \times (c - a)/2 = (b/2)^2$ (Eq. 1)

The right side of Equation 1 is still a perfect square. the simplest way to get the left side to be a perfect square is to set (c + a)/2 and (c - a)/2 each to perfect squares. For example, suppose $(c + a)/2 = 5^2 = 25$ and $(c - a)/2 = 4^2 = 16$.

From the equation,

$(b/2)^2 = (c + a)/2 \times (c - a)/2$

$= 25 \times 16$

$b^2 = 4 \times 25 \times 16 = 1600$

$b = 40$

For a and c, we have two equations in two unknowns:

$c/2 + a/2 = 25$ (Eq. 2)

$c/2 – a/2 = 16$ (Eq. 3)

Adding and subtracting Equations 2 and 3 yields:

$c = 25 + 16 = 41$ and

$a = 25 – 16 = 9$

The Pythagorean triplet is then (9, 40, 41).

General Case:

For any choice of integers m and n where $n< m$, we can get a Pythagorean triple by setting

$(c + a)/2 = m^2$ and

$(c - a)/2 = n^2$

Proceeding as we did above, we get

$(b/2)^2 = m^2 \times n^2$

$b^2/4 = m^2 \times n^2$

$b^2 = 4 m^2 \times n^2$

ISSN 1533-9211
DOI 10.17605/OSF.IO/PFZTR

$b = 2mn$

$c/2 + a/2 = m^2$

$c/2 - a/2 = n^2$

Adding and subtracting these equations gives:

$c = m^2 + n^2$

$a = m^2 - n^2$

Furthermore, we can directly substitute the above values for a, b and c in

$c^2 = a^2 + b^2$

to show that

$(m^2 + n^2)^2 = (m^2 – n^2)^2 + (2mn)^2$.

We have found a way of generating infinitely many Pythagorean triples.

Completion of Proof:

Remember that what we want are Pythagorean triplets with no common factor. What will be shown is that, we can generate all such triples by adding further restrictions on m and n that one of then must be even and the other odd and that m and n have no common factor. The proof falls into two parts:

1. Show that for m and n with one even and the other odd and having no common factor which means that the values $(m^2 + n^2)$, $(m^2 – n^2)$ and $(2mn)$ have no common factor.
2. Show that for any other solution of $c^2 = a^2 + b^2$, where a, b and c have a common factor.

Proof that m and n have no common factor:

The key to this proof is the fact that we can go backwards from the values for a, b and c to the original choice of values for m and n.

From the previous equations, we get:

$m^2 + n^2 = c$

$m^2 - n^2 = a$

Adding and subtracting these equations we get the original conditions:

$m^2 = c/2 + a/2$

$n^2 = c/2 - a/2$

ISSN 1533-9211

DOI 10.17605/OSF.IO/PFZTR

Proof by contradiction:

Now suppose that a and c have a common prime divisor p excludes 2, c = pr and a = ps for some r and s. Substituting these values in the above equations gives:

$$m^2 = (p \times (r + s))/2$$

$$n^2 = (p \times (r - s))/2$$

Implying that m^2 and n^2 have a common factor of p which contradicts the condition that m^2 and n^2 must be even and the other odd. We have shown that our method of choosing Pythagorean triples will never have a common factor.

Proof that any other Pythagorean triples must have a common factor:

(c/2 + a/2) x (c/2 – a/2) has to be a perfect square. We have covered the case where (c/2 + a/2) and (c/2 – a/2) are each perfect square. Let us consider the following cases. If (c/2 + a/2) is not a perfect square then it must have some prime factor raised to an odd numbered power. Suppose for example that (c/2 + a/2) is divisible by 7^3. Since the product of (c/2 + a/2) and (c/2 – a/2) must be perfect square this means that (c/2 – a/2) must also be divisible by 7 raised to an odd power, so that both (c/2 + a/2) and (c/2 – a/2) are divisible by 7 and in general (c/2 + a/2) and (c/2 – a/2) will both be divisible by some prime p. We then have

$$(c/2 + a/2) = p \times r \text{ and}$$

$$(c/2 - a/2) = p \times s$$

for some numbers r and s.

Adding and subtracting these equations gives

$$c = p \times (r + s) \text{ and}$$

$$a = p \times (r - s)$$

giving c and a, a common factor of p, as was to be proved.

RESULTS

Based on this investigation, the salient findings are as follows:

1. The Pythagorean triples was derived from the Pythagorean Theorem which states that "the square of the hypotenuse of a right triangle is equal to the sum of the squares on the two legs" or $a^2 + b^2 = c^2$.
2. The Pythagorean Triple is a triple of positive integers a, b, c such that $a^2 + b^2 = c^2$. The numbers can represent the sides of right-angle triangle a, b and c the sides of the triple, a and b the legs, c the hypotenuse.
3. For any choice of integers m and n where n < m, we can get a Pythagorean triple by setting values for a, b and c to show that $(m^2 + n^2)^2 = (m^2 – n^2)^2 + (2mn)^2$. This formula generates infinitely many Pythagorean triples.

The Seybold Report
ISSN 1533-9211
DOI 10.17605/OSF.IO/PFZTR

4. If restrictions on m and n that one of them must be even and the other is odd then, the values of $(m^2 + n^2)$, $(m^2 – n^2)$ and (2mn) have no common factor.

CONCLUSION

Based on the findings, the researchers arrived to the conclusion that:

For any choice of integers m and n where $n < m$, we can get a Pythagorean triple by setting

$$(c + a)/2 = m^2 \text{ and } (c – a)/2 = n^2.$$

Adding and subtracting these equations gives:

$$c = m^2 + n^2 \text{ and } a = m^2 – n^2 \text{ and } b = 2mn$$

Furthermore, we can directly substitute the above values for a, b and c in

$$c^2 = a^2 + b^2$$

to show that

$$(m^2 + n^2)^2 = (m^2 – n^2)^2 + (2mn)^2$$

which generates infinitely many Pythagorean triples.

RECOMMENDATIONS

Based on the findings and conclusions, the following suggestions to make this study more meaningful and could be used as an enrichment activity in teaching Number Theory are hereby recommended:

1. Future researchers may explore some of the properties of primitive triples. They may also count the number of triples of a given hypotenuse and fid all triples with certain restrictions. Using a programmable calculator or computer program, generate a systematic list of solutions to $x^2 + y^2 = z^2$.
2. Teachers should possess the high level of mathematical intellect, must be a well-mannered, and be known and reflected upon for their honesty, integrity and competence.
3. Mathematics teachers should always update themselves in the new trends and curriculum to cope with the fast changing world of mathematics like enrolling in the graduate school, attending seminars, trainings and symposia.

References

1. Amato, R. (2017). A characterization of Pythagorean Triples. JP Journal of Algebra, Number Theory and Applications, 39(2), 221.
2. Benett, Dan (1995). “Pythagoras Plugged In: Proofs and Problems for the Geometer’s Sketchpad” Berkeley, CA: Key Curriculum Press.

ISSN 1533-9211

DOI 10.17605/OSF.IO/PFZTR

3. Bogolomy, Alexander.(2010). Pythagorean Theorem. Interactive Mathematics Miscellany and Puzzles. New York, 2010.
4. Evans, Chris (1995). “Pythagorean Triples”. Mathematical Gazette 75, page 317.
5. Eynden, Charles V. (2006). Elementary Number Theory, 2nd Edition, McGraw-Hill Book Co. Inc., New York.
6. Overmars, A., Ntogramatzidis, L., & Venkatraman, S. (2019). A new approach to generate all Pythagorean triples. AIMS Mathematics, 4(2), 242-253.
7. Sierpinski, Waclaw (2003). Pythagorean Triangles. (Mineola, NY, Dover Publications, Inc.
8. Spencer, Donald (1995). Exploring Number Theory with Microcomputers. Camelot Publishing Company, Osmond Beach Florida, 3rd Edition.
9. Weissten, Eric W. (2008). “Pythagorean Theorem” from MathWorld—A Wolfram Web Resource

Relevant Links

1. http://www.pdffactory.com
2. http://www.mathworld.com
3. http://www.wikipedia.com
4. www.geocites.com/Charlestonmystery/triples.html
5. www.andrews.edu/calkins/mathweb.text/num//
6. www.mathmtu.edu/mathlab/courses/holt
7. www.emporia.edu.math